Problems and Solutions-VIII

P & S 8

27 December 2024

Unconstrained Optimization

Problem 17.1 (a)

Find the critical points and classify these as local max, local min, or saddle point for the function:

$$f(x, y) = x^4 + x^2 - 6xy + 3y^2.$$

Unconstrained Optimization

Solution

Calculate the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = 4x^3 + 2x - 6y, \quad \frac{\partial f}{\partial y} = -6x + 6y.$$

Set the partial derivatives to zero:

$$-6x + 6y = 0 \implies x = y.$$

Substitute $y = x$ into $\frac{\partial f}{\partial x} = 0$:

$$4x^3 + 2x - 6x = 4x^3 - 4x = 0 \implies x(x^2 - 1) = 0.$$

Solutions are:

$$x = \pm 1, 0 \quad \text{and} \quad y = \pm 1, 0.$$

Thus, the critical points are:

$$(-1, -1), (0, 0), (1, 1).$$

Unconstrained Optimization

Hessian Matrix

The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2, \quad \frac{\partial^2 f}{\partial y^2} = 6, \quad \frac{\partial^2 f}{\partial x \partial y} = -6.$$

The Hessian matrix is:

$$H(x, y) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

Classification at $(-1, -1)$

$$H(-1, -1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}.$$

1st LPM: 14. **2nd LPM**: $\det(H(-1, -1)) = (14)(6) - (-6)(-6) = 60$. The Hessian is **positive definite**, so $(-1, -1)$ is a **local minimum**.

Unconstrained Optimization

Classification at (1,1)

$$H(1,1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}.$$

1st LPM: 14. **2nd LPM**: $\det(H(1,1)) = (14)(6) - (-6)(-6) = 60$. The Hessian is **positive definite**, so $(1,1)$ is a **local minimum**.

Classification at $(0,0)$

$$H(0,0) = \begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

1st LPM: 2. **2nd LPM**: $\det(H(0,0)) = (2)(6) - (-6)(-6) = -28$. The Hessian is **indefinite**, so $(0,0)$ is a **saddle point**.

Unconstrained Optimization

Problem 17.1 (b)

Find the critical points and classify these as local max, local min, or saddle point for the function:

$$f(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y + 5.$$

Unconstrained Optimization

Solution

Calculate the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = 2x - 6y + 10, \quad \frac{\partial f}{\partial y} = -6x + 4y + 2.$$

Set the partial derivatives to zero:

$$2x - 6y + 10 = 0 \quad \text{and} \quad -6x + 4y + 2 = 0.$$

From the first equation:

$$2x - 6y = -10 \implies x = 3y - 5.$$

Substitute $x = 3y - 5$ into the second equation:

$$-6(3y-5)+4y+2=0 \implies -18y+30+4y+2=0 \implies -14y+32=0 \implies y=\frac{16}{7}.$$

Unconstrained Optimization

Solution

Substitute $y = \frac{16}{7}$ into $x = 3y - 5$:

$$x = 3\left(\frac{16}{7}\right) - 5 = \frac{48}{7} - 5 = \frac{13}{7}.$$

Thus, the critical point is:

$$\left(\frac{13}{7}, \frac{16}{7}\right).$$

Unconstrained Optimization

Hessian Matrix

The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = -6.$$

The Hessian matrix is:

$$H(x, y) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}.$$

Classification at $\left(\frac{13}{7}, \frac{16}{7}\right)$

Compute the determinant of the Hessian:

$$\det(H) = (2)(4) - (-6)(-6) = 8 - 36 = -28.$$

Since the determinant is negative, the Hessian is **indefinite**, and the critical point $\left(\frac{13}{7}, \frac{16}{7}\right)$ is a **saddle point**.

Unconstrained Optimization

Problem 17.1 (d)

Find the critical points and classify these as local max, local min, or saddle point for the function:

$$f(x, y) = 3x^4 + 3x^2y - y^3.$$

Unconstrained Optimization Solution

Solution to Problem 17.1 (d)

$$f(x,y) = 3x^4 + 3x^2y - y^3$$

First Order Partial Derivatives

$$f_x = 12x^3 + 6xy, \quad f_y = 3x^2 - 3y^2$$

Solve for Critical Points ($f_x = 0, f_y = 0$)

From $f_y = 0$, we get:

$$3x^2 - 3y^2 = 0 \implies x^2 = y^2 \implies y = \pm x.$$

Unconstrained Optimization Solution

Solution to Problem 17.1 (d)

From $f_x = 0$, we get:

$$12x^3 + 6xy = 6x(2x^2 + y) = 0.$$

- If $x = 0$, then $y = 0$. Critical point: $(0, 0)$.
- If $2x^2 + y = 0$, substituting $y = -2x^2$ into $y = \pm x$:
  - If $y = x$, $-2x^2 = x \implies x(2x+1) = 0 \implies x = 0$ or $x = -\frac{1}{2}$.
    Critical point: $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.
  - If $y = -x$, $-2x^2 = -x \implies x(2x-1) = 0 \implies x = 0$ or $x = \frac{1}{2}$.
    Critical point: $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Critical Points:

$$(0, 0), \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right).$$

Unconstrained Optimization Solution

Solution to Problem 17.1 (d)

Second Order Partial Derivatives and Hessian

$$f_{xx} = 36x^2 + 6y, \quad f_{yy} = -6y, \quad f_{xy} = f_{yx} = 6x.$$

The Hessian matrix is:

$$H(x, y) = \begin{bmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{bmatrix}.$$

Classify Critical Points

- At $(0, 0)$:

$$H(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Unconstrained Optimization Solution

Solution to Problem 17.1 (d)

Determinant = 0, Hessian is indeterminate. Since $f(0, y) = -y^3$, $(0, 0)$ is a **saddle point**.

- At $\left(-\frac{1}{2}, -\frac{1}{2}\right)$:

  $$H\left(-\frac{1}{2}, -\frac{1}{2}\right) = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}.$$

  Determinant = $9 > 0$, trace = $9 > 0$, Hessian is **positive definite.** $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ is a **local minimum**.

- At $\left(\frac{1}{2}, -\frac{1}{2}\right)$:

  $$H\left(\frac{1}{2}, -\frac{1}{2}\right) = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}.$$

  Determinant = $9 > 0$, trace = $9 > 0$, Hessian is **positive definite.** $\left(\frac{1}{2}, -\frac{1}{2}\right)$ is a **local minimum**.

Unconstrained Optimization Solution

Solution to Problem 17.1 (d)

Conclusion:

- $(0, 0)$: Saddle point.
- $\left(-\frac{1}{2}, -\frac{1}{2}\right)$: Local minimum.
- $\left(\frac{1}{2}, -\frac{1}{2}\right)$: Local minimum.