Problems and Solutions-VII

P & S 7

20 December 2024

Partial & Total Derivative

Problem 14.1 (a)

Compute all the partial derivatives

$$f(x, y) = 4x^2y - 3xy^3 + 6x$$

Partial & Total Derivative

Solution

Calculate the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(4x^2y - 3xy^3 + 6x) = 8xy - 3y^3 + 6,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(4x^2y - 3xy^3 + 6x) = 4x^2 - 9xy^2.$$

Thus, the first-order partial derivatives are:

$$\frac{\partial f}{\partial x} = 8xy - 3y^3 + 6, \quad \frac{\partial f}{\partial y} = 4x^2 - 9xy^2.$$

Partial & Total Derivative

Problem 14.1 (d)

Compute all the partial derivatives

$$f(x, y) = e^{2x+3y}$$

Partial & Total Derivative

Solution

Calculate the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{2x+3y}\right).$$

Using the chain rule:

$$\frac{\partial f}{\partial x} = e^{2x+3y} \cdot \frac{\partial}{\partial x}(2x+3y) = e^{2x+3y} \cdot 2 = 2e^{2x+3y}.$$

Similarly, for $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{2x+3y}\right).$$

Using the chain rule:

$$\frac{\partial f}{\partial y} = e^{2x+3y} \cdot \frac{\partial}{\partial y}(2x+3y) = e^{2x+3y} \cdot 3 = 3e^{2x+3y}.$$

Partial & Total Derivative

Solution

Thus, the first-order partial derivatives are:

$$\frac{\partial f}{\partial x} = 2e^{2x+3y}, \quad \frac{\partial f}{\partial y} = 3e^{2x+3y}.$$

Partial & Total Derivative

Problem 14.1 (e)

Compute all the partial derivatives

$$f(x, y) = \frac{x + y}{x - y}$$

Partial & Total Derivative

Solution

Calculate the first-order partial derivatives using the quotient rule:

$$\frac{\partial f}{\partial x} = \frac{(x-y) \cdot \frac{\partial}{\partial x}(x+y) - (x+y) \cdot \frac{\partial}{\partial x}(x-y)}{(x-y)^2}.$$

Simplify:

$$\frac{\partial f}{\partial x} = \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{(x-y)^2}.$$

For $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = \frac{(x-y) \cdot \frac{\partial}{\partial y}(x+y) - (x+y) \cdot \frac{\partial}{\partial y}(x-y)}{(x-y)^2}.$$

Simplify:

$$\frac{\partial f}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \frac{x-y+x+y}{(x-y)^2} = \frac{2x}{(x-y)^2}.$$

Partial & Total Derivative

Solution

Thus, the first-order partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{-2y}{(x-y)^2}, \quad \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}.$$

Marginal Product

Example 1: Production Function

Consider the production function:

$$Q = 5K^{1/2}L^{1/3}.$$

When $K = 64$ and $L = 27$, the output Q is:

$$Q = 5 \cdot 64^{1/2} \cdot 27^{1/3} = 5 \cdot 8 \cdot 3 = 120.$$

Calculating the partial derivatives:

$$\frac{\partial Q}{\partial K} = \left(5L^{1/3}\right) \cdot \frac{1}{2}K^{-1/2} = \frac{5L^{1/3}}{2K^{1/2}},$$

treating L as constant, and:

$$\frac{\partial Q}{\partial L} = \left(5K^{1/2}\right) \cdot \frac{1}{3}L^{-2/3} = \frac{5K^{1/2}}{3L^{2/3}},$$

treating K as constant.

Marginal Product

Example: Production Function (cont.)

Furthermore:

$$\frac{\partial Q}{\partial K}(64,27)=\frac{5\cdot 27^{1/3}}{2\cdot 64^{1/2}}=\frac{5\cdot 3}{2\cdot 8}=\frac{15}{16},$$

and:

$$\frac{\partial Q}{\partial L}(64,27)=\frac{5\cdot 64^{1/2}}{3\cdot 27^{2/3}}=\frac{5\cdot 8}{3\cdot 9}=\frac{40}{27}.$$

If L is held constant and K increases by ΔK, Q will increase approximately by

$\frac{15}{16}\cdot\Delta K.$

For an increase in K of 4 units, we use this approximation to estimate $Q(68,27)$ as:

$$120+\frac{15}{16}\cdot 4=120+3.75=123.75.$$

Similarly, a 3-unit decrease in L induces a $3\cdot\frac{40}{27}=\frac{120}{27}\approx 4.44$-unit decrease in Q, so:

$$Q(64,24)=120-4.44=115.56.$$

Marginal Product

Example 2: Linear Production Function

Consider the linear production function:

$$Q = 3K + 2L.$$

When $K = 10$ and $L = 5$, the output Q is:

$$Q = 3 \cdot 10 + 2 \cdot 5 = 30 + 10 = 40.$$

Calculating the partial derivatives:

$$\frac{\partial Q}{\partial K} = 3, \quad \frac{\partial Q}{\partial L} = 2.$$

The marginal product of capital is constant at 3, and the marginal product of labor is constant at 2.

Marginal Product

Example 2: Linear Production Function (cont.)

Furthermore:

$$\frac{\partial Q}{\partial K}(10,5) = 3,$$

and:

$$\frac{\partial Q}{\partial L}(10,5) = 2.$$

If L is held constant and K increases by ΔK, Q will increase approximately by $3 \cdot \Delta K$.

For an increase in K of 4 units, we use this approximation to estimate $Q(14,5)$ as:

$$40 + 3 \cdot 4 = 40 + 12 = 52.$$

Similarly, a 2-unit increase in L induces a $2 \cdot 2 = 4$-unit increase in Q, so:

$$Q(10,7) = 40 + 4 = 44.$$

Marginal Product

Problem 14.2

Consider the Cobb-Douglas production function:

$$q = kx_1^{a_1}x_2^{a_2},$$

where k, a_1, and a_2 are constants. The partial derivatives are:

$$\frac{\partial q}{\partial x_1} = \frac{\partial}{\partial x_1}\left(kx_1^{a_1}x_2^{a_2}\right) = ka_1x_1^{a_1-1}x_2^{a_2},$$

treating x_2 as constant, and:

$$\frac{\partial q}{\partial x_2} = \frac{\partial}{\partial x_2}\left(kx_1^{a_1}x_2^{a_2}\right) = ka_2x_1^{a_1}x_2^{a_2-1},$$

treating x_1 as constant.