Problems and Solutions-VI

P & S 6

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Determinants

Problem 9.11 (a)

Use Theorem 9.4 to invert the following matrix:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$$

Determinants and Inverse Calculation

Solution

Calculate the Determinant and Cofactors

The determinant of A is calculated as:

$$\det(A) = (4)(1) - (3)(1) = 4 - 3 = 1.$$

Since $\det(A) \neq 0$, the matrix A is invertible.

To find the cofactors, we calculate the determinant of the 1×1 minors for each element of A, with alternating signs.

$$\text{Cofactor Matrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where:

$$C_{11} = +\left|1\right| = 1, \quad C_{12} = -\left|1\right| = -1,$$
$$C_{21} = -\left|3\right| = -3, \quad C_{22} = +\left|4\right| = 4.$$

Thus, the cofactor matrix is:

$$\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}.$$

Determinants and Inverse Calculation

Solution (cont.)

The adjoint of A is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

From Theorem 9.4:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A).$$

Substitute $\det(A) = 1$ and $\text{adj}(A) = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$:

$$A^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

Determinants

Problem 9.11 (b)

Use Theorem 9.4 to invert the following matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Determinants and Inverse Calculation

Solution

Calculate the Determinant of B

The determinant of B is:

$$\det(B) = 1 \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix}.$$

Compute the minors:

$$\begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} = (5)(8) - (6)(0) = 40, \quad \begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} = (0)(8) - (6)(1) = -6, \quad \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} = (0)(0) - (5)(1) =$$

Substitute back:

$$\det(B) = 1(40) - 2(-6) + 3(-5) = 40 + 12 - 15 = 37.$$

Since $\det(B) \neq 0$, the matrix B is invertible.

Determinants and Inverse Calculation

Solution (cont.)

Calculate cofactors

The cofactor for each element B_{ij} is calculated by removing the i-th row and j-th column and taking the determinant of the resulting minor, with alternating signs.

$$\text{Cofactor Matrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where:

$$C_{11} = +\begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} = 40, \quad C_{12} = -\begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} = 6, \quad C_{13} = +\begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} = -5,$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = -16, \quad C_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} = 5, \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2,$$

$$C_{31} = +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 8 \end{vmatrix} = -8, \quad C_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5.$$

Determinants and Inverse Calculation

Solution (cont.)

Calculate cofactors Thus, the cofactor matrix is:

$$\begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & -2 \\ -3 & -8 & 5 \end{pmatrix}.$$

The adjoint of B is the transpose of the cofactor matrix:

$$\text{adj}(B) = \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -8 \\ -5 & -2 & 5 \end{pmatrix}.$$

Determinants and Inverse Calculation

Solution (cont.)

From Theorem 9.4:

$$B^{-1} = \frac{1}{\text{det}(B)} \cdot \text{adj}(B).$$

Substitute $\text{det}(B) = 37$ and $\text{adj}(B)$:

$$B^{-1} = \frac{1}{37} \cdot \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -8 \\ -5 & -2 & 5 \end{pmatrix}.$$

The final inverse is:

$$B^{-1} = \begin{pmatrix} \frac{40}{37} & -\frac{16}{37} & -\frac{3}{37} \\ \frac{6}{37} & \frac{5}{37} & -\frac{8}{37} \\ -\frac{5}{37} & -\frac{2}{37} & \frac{5}{37} \end{pmatrix}.$$

Determinants

Problem 9.11 (c)

Use Theorem 9.4 to invert the following matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Determinants and Inverse Calculation

Solution

Calculate the Determinant of A

The determinant of A is:

$$\det(A) = (a)(d) - (b)(c).$$

Since $\det(A) \neq 0$, the matrix A is invertible.

Calculate the Cofactors

For a 2×2 matrix, the cofactors are:

$$\text{Cofactor Matrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where:

$$C_{11} = d, \quad C_{12} = c, \quad C_{21} = -b, \quad C_{22} = a.$$

Thus, the cofactor matrix is:

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Determinants and Inverse Calculation

Compute the Adjoint Matrix and Inverse

The adjoint of A is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

From Theorem 9.4:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A).$$

Substitute $\det(A) = (a)(d) - (b)(c)$ and $\text{adj}(A) = \begin{pmatrix} d & -b \\ -d & a \end{pmatrix}$:

$$A^{-1} = \frac{1}{(a)(d) - (b)(c)} \cdot \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

The final inverse is:

$$A^{-1} = \begin{pmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-bc)} \\ \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{pmatrix}.$$

Determinants

Problem 9.13 (a)

Use Cramer's rule to solve the following equation system: $5x_1 + x_2 = 3$,

$$2x_1 - x_2 = 4.$$

Determinants

Solution

We can use Cramer's Rule to solve the given system of equations, written in matrix form as:

$$\begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The determinant of the coefficient matrix A is:

$$A = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix}, \quad \text{det}(A) = (5)(-1) - (2)(1) = -5 - 2 = -7.$$

The matrix A_1 is formed by replacing the first column of A with the column vector $\mathbf{b}$:

$$A_1 = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}.$$

The determinant of A_1 is:

$$\text{det}(A_1) = (3)(-1) - (4)(1) = -3 - 4 = -7.$$

Determinants

Solution (cont.)

The matrix A_2 is formed by replacing the second column of A with the column vector $\mathbf{b}$:

$$A_2 = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}.$$

The determinant of A_2 is:

$$\text{det}(A_2) = (5)(4) - (2)(3) = 20 - 6 = 14.$$

Using Cramer's Rule:

$$x_1 = \frac{\text{det}(A_1)}{\text{det}(A)} = \frac{-7}{-7} = 1, \quad x_2 = \frac{\text{det}(A_2)}{\text{det}(A)} = \frac{14}{-7} = -2.$$

Thus, the solution is:

$$x_1 = 1, \quad x_2 = -2.$$

Determinants

Problem 9.13 (b)

Use Cramer's Rule to solve the following equation system:

$$\begin{aligned} 2x_1 - 3x_2 &= 2, \\ 4x_1 - 6x_2 + x_3 &= 7, \\ x_1 + 10x_2 &= 1. \end{aligned}$$

Determinants

Solution

We can use Cramer's Rule to solve the given system of equations, written in matrix form as:

$$\begin{pmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}.$$

The determinant of the coefficient matrix A is:

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{pmatrix}.$$

Using cofactor expansion along the first row:

$$\text{det}(A) = 2 \begin{vmatrix} -6 & 1 \\ 10 & 0 \end{vmatrix} - (-3) \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix}.$$

Calculate the minors:

$$\begin{vmatrix} -6 & 1 \\ 10 & 0 \end{vmatrix} = (-6)(0) - (10)(1) = -10, \quad \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = (4)(0) - (1)(1) = -1.$$

Determinants

Solution (cont.)

To find x_1, replace the first column of A with the column vector $\mathbf{b}$:

$$A_1 = \begin{pmatrix} 2 & -3 & 0 \\ 7 & -6 & 1 \\ 1 & 10 & 0 \end{pmatrix}.$$

To find x_2, replace the second column of A with $\mathbf{b}$:

$$A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 7 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

To find x_3, replace the third column of A with $\mathbf{b}$:

$$A_3 = \begin{pmatrix} 2 & -3 & 2 \\ 4 & -6 & 7 \\ 1 & 10 & 1 \end{pmatrix}.$$

Determinants

Solution (cont.)

Using cofactor expansion along the first row:

$$\det(A_1) = 2(-10) - (-3)(-1) = -20 - 3 = -23.$$

Using cofactor expansion along the first row:

$$\det(A_2) = 2(-1) - 2(-1) = -2 + 2 = 0.$$

Using cofactor expansion along the first row:

$$\det(A_3) = 2(-76) - (-3)(-3) + 2(46) = -152 - 9 + 92 = -69.$$

Determinants

Solution (cont.)

Calculate each determinant:

$$\det(A_1) = -23, \quad \det(A_2) = 0, \quad \det(A_3) = -69.$$

Using Cramer's Rule:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-23}{-23} = 1, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{0}{-23} = 0, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-69}{-23} = 3.$$

Thus, the solution is:

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 3.$$