Problems and Solutions-V

P & S 5

25 October 2024

Example 5.8 (a)

Problem

Compute the first and second derivatives of $f(x) = xe^{3x}$.

First Derivative of $f(x) = xe^{3x}$

1) Use the product rule:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

where:

$$u(x) = x \quad \text{and} \quad v(x) = e^{3x}.$$

2) Compute the derivatives of $u(x)$ and $v(x)$:

$$u'(x) = 1$$

$$v'(x) = 3e^{3x} \quad \text{(by the chain rule).}$$

Example 5.8 (a)

First Derivative of $f(x) = xe^{3x}$

3) Apply the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Substituting the derivatives of $u(x)$ and $v(x)$:

$$f'(x) = 1 \cdot e^{3x} + x \cdot 3e^{3x}.$$

4) Simplify the expression:

$$f'(x) = e^{3x} + 3xe^{3x}.$$

Factor out e^{3x}:

$$f'(x) = e^{3x}(1 + 3x).$$

Thus, the first derivative is:

$$f'(x) = e^{3x}(1 + 3x).$$

Example 5.8 (a)

Second Derivative of $f(x) = xe^{3x}$

Compute the second derivative of $f(x) = xe^{3x}$.
1) Start with the first derivative:

$$f'(x) = e^{3x}(1 + 3x).$$

2) Use the product rule again. Let:

$$u(x) = e^{3x} \quad \text{and} \quad v(x) = 1 + 3x.$$

Example 5.8 (a)

Compute Derivatives of $u(x)$ and $v(x)$

3) Compute the derivatives:

$$u'(x) = 3e^{3x}$$

$$v'(x) = 3.$$

4) Apply the product rule:

$$f''(x) = u'(x)v(x) + u(x)v'(x)$$

Substituting the derivatives:

$$f''(x) = 3e^{3x}(1 + 3x) + e^{3x} \cdot 3.$$

Example 5.8 (a)

Second Derivative of $f(x) = xe^{3x}$

5) Simplify the expression:

$$f''(x) = 3e^{3x}(1 + 3x) + 3e^{3x}.$$

Factor out $3e^{3x}$:

$$f''(x) = 3e^{3x}(2 + 3x).$$

Thus, the second derivative is:

$$f''(x) = 3e^{3x}(2 + 3x).$$

Example 5.8 (b)

Problem

Compute the first and second derivatives of $f(x) = e^{x^2+3x-2}$.

First Derivative of $f(x) = e^{x^2+3x-2}$

1) Use the chain rule for the derivative of the exponential function $e^{g(x)}$, which states:

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$$

The inside function is $g(x) = x^2 + 3x - 2$.

Example 5.8 (b)

First Derivative of $f(x) = e^{x^2+3x-2}$

2) Compute the derivative of the inside function $g(x) = x^2 + 3x - 2$:

$$g'(x) = 2x + 3.$$

3) Apply the chain rule:

$$f'(x) = e^{x^2+3x-2} \cdot (2x + 3).$$

Thus, the first derivative is:

$$f'(x) = (2x + 3)e^{x^2+3x-2}.$$

Example 5.8 (b)

Second Derivative of $f(x) = e^{x^2+3x-2}$

Compute the second derivative of $f(x) = e^{x^2+3x-2}$.
1) Start with the first derivative:

$$f'(x) = (2x+3)e^{x^2+3x-2}.$$

2) Use the product rule to differentiate $f'(x) = (2x+3)e^{x^2+3x-2}$.
The product rule states:

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$$

Let:

$$u(x) = (2x+3) \quad \text{and} \quad v(x) = e^{x^2+3x-2}.$$

Example 5.8 (b)

Second Derivative of $f(x) = e^{x^2+3x-2}$

3) Compute the derivatives of $u(x) = 2x + 3$ and $v(x) = e^{x^2+3x-2}$:

$$u'(x) = 2.$$

To compute $v'(x)$, use the chain rule:

$$v'(x) = e^{x^2+3x-2} \cdot (2x + 3).$$

4) Apply the product rule:

$$f''(x) = u'(x)v(x) + u(x)v'(x)$$

Substituting the derivatives:

$$f''(x) = 2 \cdot e^{x^2+3x-2} + (2x + 3) \cdot e^{x^2+3x-2} \cdot (2x + 3).$$

Example 5.8 (b)

Second Derivative of $f(x) = e^{x^2+3x-2}$

5) Simplify the expression:

$$f''(x) = e^{x^2+3x-2}\left(2 + (2x+3)^2\right).$$

Expanding $(2x+3)^2$:

$$f''(x) = e^{x^2+3x-2}\left(2 + (4x^2+12x+9)\right).$$

$$f''(x) = e^{x^2+3x-2}\left(4x^2+12x+11\right).$$

Thus, the second derivative is:

$$f''(x) = e^{x^2+3x-2}\left(4x^2+12x+11\right).$$

Example 5.8 (c)

Problem

Compute the first and second derivatives of $f(x) = \ln\left((x^4+2)^2\right)$.

First Derivative of $f(x) = \ln\left((x^4+2)^2\right)$

1) Use the logarithmic identity $\ln(a^b) = b\ln(a)$ to simplify:

$$f(x) = 2\ln(x^4+2)$$

2) Differentiate using the chain rule:

$$f'(x) = 2 \cdot \frac{1}{x^4+2} \cdot \frac{d}{dx}(x^4+2)$$

$$f'(x) = 2 \cdot \frac{1}{x^4+2} \cdot 4x^3$$

Thus, the first derivative is:

$$f'(x) = \frac{8x^3}{x^4+2}$$

Example 5.8 (c)

Second Derivative of $f(x) = \ln\left((x^4 + 2)^2\right)$

1) Start with the first derivative:

$$f'(x) = \frac{8x^3}{x^4 + 2}$$

2) Use the quotient rule to differentiate:

$$f''(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

where $u(x) = 8x^3$ and $v(x) = x^4 + 2$.

Example 5.8 (c)

Second Derivative of $f(x) = \ln\left((x^4+2)^2\right)$

3) Compute the derivatives of $u(x) = 8x^3$ and $v(x) = x^4 + 2$:

$$u'(x) = 24x^2 \quad \text{and} \quad v'(x) = 4x^3$$

4) Apply the quotient rule:

$$f''(x) = \frac{24x^2(x^4+2) - 8x^3(4x^3)}{(x^4+2)^2}$$

Simplify the numerator:

$$f''(x) = \frac{24x^6 + 48x^2 - 32x^6}{(x^4+2)^2}$$

$$f''(x) = \frac{-8x^6 + 48x^2}{(x^4+2)^2}$$

Thus, the second derivative is:

$$f''(x) = \frac{-8x^6 + 48x^2}{(x^4+2)^2}$$

Example 5.8 (d)

Problem

Compute the first and second derivatives of $f(x) = \frac{x}{e^x}$.

First Derivative of $f(x) = \frac{x}{e^x}$

1) Use the quotient rule:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

where:

$$u(x) = x \quad \text{and} \quad v(x) = e^x$$

2) Compute the derivatives of $u(x)$ and $v(x)$:

$$u'(x) = 1 \quad \text{and} \quad v'(x) = e^x$$

3) Apply the quotient rule:

$$f'(x) = \frac{e^x - xe^x}{e^{2x}}$$

Simplify the expression:

Example 5.8 (d)

Second Derivative of $f(x) = \frac{x}{e^x}$

1: Start with the first derivative)

$$f'(x) = \frac{1-x}{e^x}$$

2: Use the quotient rule again to differentiate $f'(x)$:

$$f''(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

where:

$$u(x) = 1 - x \quad \text{and} \quad v(x) = e^x$$

Example 5.8 (d)

Second Derivative of $f(x) = \frac{x}{e^x}$

3: Compute the derivatives of $u(x) = 1 - x$ and $v(x) = e^x$)

$$u'(x) = -1 \quad \text{and} \quad v'(x) = e^x$$

4: Apply the quotient rule)

$$f''(x) = \frac{(-1) \cdot e^x - (1 - x) \cdot e^x}{(e^x)^2}$$

Simplify the expression:

$$f''(x) = \frac{-e^x - (1 - x)e^x}{e^{2x}}$$

Factor out e^x:

$$f''(x) = \frac{e^x\left(-1 - (1 - x)\right)}{e^{2x}} = \frac{e^x(x - 2)}{e^{2x}}$$

Remove e^x from the numerator and denominator:

$$f''(x) = \frac{x - 2}{e^x}$$

Example 5.8 (e)

Problem

Compute the first and second derivatives of $f(x) = \dfrac{x}{\ln x}$.

First Derivative of $f(x) = \dfrac{x}{\ln x}$

1) Use the quotient rule:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

where:

$$u(x) = x \quad \text{and} \quad v(x) = \ln x$$

2) Compute the derivatives of $u(x)$ and $v(x)$:

$$u'(x) = 1 \quad \text{and} \quad v'(x) = \frac{1}{x}$$

Example 5.8 (e)

First Derivative of $f(x) = \dfrac{x}{\ln x}$

3) Apply the quotient rule:

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}$$

Thus, the first derivative is:

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}$$

Example 5.8 (e)

Second Derivative of $f(x) = \dfrac{x}{\ln x}$

1) Start with the first derivative:

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}$$

2) Use the quotient rule again to differentiate $f'(x)$:

$$f''(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

where:

$$u(x) = \ln x - 1 \quad \text{and} \quad v(x) = (\ln x)^2$$

Example 5.8 (e)

Second Derivative of $f(x) = \frac{x}{\ln x}$

3) Compute the derivatives of $u(x) = \ln x - 1$ and $v(x) = (\ln x)^2$:

$$u'(x) = \frac{1}{x}, \quad v'(x) = \frac{2 \ln x}{x}$$

4) Apply the quotient rule:

$$f''(x) = \frac{\frac{1}{x}(\ln x)^2 - (\ln x - 1) \cdot \frac{2 \ln x}{x}}{(\ln x)^4}$$

Example 5.8 (e)

Second Derivative of $f(x) = \dfrac{x}{\ln x}$

5) Simplify the expression:

Factor out $\dfrac{1}{x}$ from the numerator:

$$f''(x) = \frac{\frac{1}{x}\left((\ln x)^2 - 2\ln x(\ln x - 1)\right)}{(\ln x)^4}$$

Simplify inside the parentheses:

$$f''(x) = \frac{\frac{1}{x}\left(-(\ln x)^2 + 2\ln x\right)}{(\ln x)^4}$$

Thus, the second derivative is:

$$f''(x) = \frac{-(\ln x)^2 + 2\ln x}{x(\ln x)^4}$$