Problems and Solutions-IV

P & S 4

18 October 2024

Example 4.1 (a)

Problem

- Consider the functions $g(x) = x^2 + 4$ and $h(z) = 5z - 1$.
- Find the composite function $(g \circ h)(z)$.
- Write out the function and simplify.

Solution

Solution

The composite function $(g \circ h)(z)$ is formed by substituting $h(z)$ into $g(x)$:

$$(g \circ h)(z) = g(h(z)) = g(5z - 1)$$

Substituting $h(z) = 5z - 1$ into $g(x) = x^2 + 4$, we get:

$$g(5z - 1) = (5z - 1)^2 + 4$$

Now, let's expand $(5z - 1)^2$:

$$(5z - 1)^2 = (5z)^2 - 2 \cdot 5z \cdot 1 + 1^2 = 25z^2 - 10z + 1$$

So the composite function is:

$$(g \circ h)(z) = 25z^2 - 10z + 1 + 4 = 25z^2 - 10z + 5$$

Example 4.3 (a)

Problem

- Consider the functions $g(x) = x^2 + 4$ and $h(z) = 5z - 1$.
- Find the derivative of the composite function $(g \circ h)(z)$.
- Use the Chain Rule to compute the derivative from the two component functions.

Solution

Solution

find $\frac{d}{dz}[g(h(z))]$. Using the Chain Rule:

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z)$$

1) Compute the derivative of the outside function $g(x) = x^2 + 4$:

$$g'(x) = 2x$$

2) Compute the derivative of the inside function $h(z) = 5z - 1$:

$$h'(z) = 5$$

Solution (cont.)

Solution

3) Apply the Chain Rule:

$$\frac{d}{dz}\left[g(h(z))\right] = g'(h(z)) \cdot h'(z)$$

Substituting $h(z) = 5z - 1$ into $g'(x)$, we get:

$$g'(h(z)) = 2(5z - 1)$$

Now, multiply by $h'(z) = 5$:

$$\frac{d}{dz}\left[g(h(z))\right] = 2(5z - 1) \cdot 5$$

4) Simplify the result:

$$\frac{d}{dz}\left[g(h(z))\right] = 10(5z - 1) = 50z - 10$$

Example 4.3 (a)

Problem

Consider the functions: $g(x) = x^2 + 4$ and $h(z) = 5z - 1$

Compute the derivative of the composite function $(g \circ h)(z)$

1. each derivative directly
2. using the Chain Rule

Simplify your Solution and compare both methods.

Solution (part 1)

Direct computation of the derivative

The composite function is:

$$(g \circ h)(z) = g(h(z)) = g(5z - 1)$$

Substitute $h(z) = 5z - 1$ into $g(x) = x^2 + 4$:

$$g(5z - 1) = (5z - 1)^2 + 4$$

Now differentiate directly with respect to z:

Differentiation

$$\frac{d}{dz}\left[(5z - 1)^2 + 4\right] = \frac{d}{dz}\left[(5z - 1)^2\right] + \frac{d}{dz}[4]$$

The derivative of the constant 4 is 0, so we focus on:

$$\frac{d}{dz}\left((5z - 1)^2\right) = 2(5z - 1) \cdot \frac{d}{dz}(5z - 1) = 2(5z - 1) \cdot 5 = 10(5z - 1)$$

Thus, the derivative is:

$$\frac{d}{dz}\left[(5z - 1)^2 + 4\right] = 10(5z - 1)$$

Solution (part 2)

Derivative Using the Chain Rule

The Chain Rule states that:

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z)$$

Let's compute each part:

- The derivative of $g(x) = x^2 + 4$ is:

$$g'(x) = 2x$$

- The derivative of $h(z) = 5z - 1$ is:

$$h'(z) = 5$$

Chain Rule Application

Now, apply the Chain Rule:

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z) = 2(5z - 1) \cdot 5 = 10(5z - 1)$$

Comparison of the Results

Both methods give the same result:

- Direct differentiation: $10(5z - 1)$
- Chain Rule application: $10(5z - 1)$

Thus, the derivative of the composite function $(g \circ h)(z)$ is:

$$\boxed{10(5z - 1)}$$

Example 4.1 (b)

Problem

- Consider the functions $g(x) = x^3$ and $h(z) = (z-1)(z+1)$.
- Find the composite function $(g \circ h)(z)$.
- Write out the function and simplify.

Solution

Solution

The composite function $(g \circ h)(z)$ is formed by substituting $h(z)$ into $g(x)$:

$$(g \circ h)(z) = g(h(z)) = g\left((z-1)(z+1)\right)$$

Substituting $h(z) = (z-1)(z+1)$ into $g(x) = x^3$, we get:

$$(g \circ h)(z) = \left((z-1)(z+1)\right)^3$$

Now, simplify $(z-1)(z+1)$:

$$(z-1)(z+1) = z^2 - 1$$

Solution (cont.)

Solution

So the composite function becomes:

$$(g \circ h)(z) = (z^2 - 1)^3$$

$$(g \circ h)(z) = (z - 1)^3 \cdot (z + 1)^3$$

$$(z - 1)^3 \cdot (z + 1)^3 = z^6 - 3z^4 + 3z^2 - 1$$

So the composite function is:

$$(g \circ h)(z) = z^6 - 3z^4 + 3z^2 - 1$$

Example 4.3 (b)

Problem

- Consider the functions $g(x) = x^3$ and $h(z) = (z-1)(z+1)$.
- Find the derivative of the composite function $(g \circ h)(z)$.
- Use the Chain Rule to compute the derivative from the two component functions.

Solution

Solution

Using the Chain Rule:

$$\frac{d}{dz}\left[g(h(z))\right] = g'(h(z)) \cdot h'(z)$$

1) Compute the derivative of the outside function $g(x) = x^3$

$$g'(x) = 3x^2$$

2) Compute the derivative of the inside function $h(z) = (z-1)(z+1)$:

First, simplify $h(z)$:

$$h(z) = (z-1)(z+1) = z^2 - 1$$

Now differentiate it:

$$h'(z) = 2z$$

Solution (cont.)

3) Apply the Chain Rule

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z)$$

Substituting $h(z) = z^2 - 1$ into $g'(x)$, we get:

$$g'(h(z)) = 3(z^2 - 1)^2$$

Now, multiply by $h'(z) = 2z$:

$$\frac{d}{dz}[g(h(z))] = 3(z^2 - 1)^2 \cdot 2z$$

4) Simplify the result

$$\frac{d}{dz}[g(h(z))] = 6z(z^2 - 1)^2$$

$$(g \circ h)'(z) = 6z(z - 1)^2 \cdot (z + 1)^2$$

Example 4.3 (b)

Problem

Consider the functions: $g(x) = x^3$ and $h(z) = (z-1)(z+1)$

Compute the derivative of the composite function $(g \circ h)(z)$

1. each derivative directly
2. using the Chain Rule

Simplify your solution and compare both methods.

Solution (part 1)

Direct computation of the derivative

The composite function is:

$$(g \circ h)(z) = g(h(z)) = g((z-1)(z+1))$$

Substitute $h(z) = (z-1)(z+1)$ into $g(x) = x^3$:

$$g((z-1)(z+1)) = [(z-1)(z+1)]^3$$

Now differentiate directly with respect to z:

Differentiation

First, simplify $(z-1)(z+1)$:

$$(z-1)(z+1) = z^2 - 1$$

Now the composite function becomes:

$$g(h(z)) = (z^2 - 1)^3$$

Solution (part 1)

Differentiation

Differentiating with respect to z:

$$\frac{d}{dz}\left[(z^2-1)^3\right] = 3(z^2-1)^2 \cdot \frac{d}{dz}(z^2-1)$$

The derivative of z^2-1 is $2z$, so we get:

$$\frac{d}{dz}\left[(z^2-1)^3\right] = 3(z^2-1)^2 \cdot 2z = 6z(z^2-1)^2$$

Thus, the derivative is:

$$\frac{d}{dz}\left[(z^2-1)^3\right] = 6z(z^2-1)^2$$

Solution (part 2)

Derivative Using the Chain Rule

The Chain Rule states that:

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z)$$

Let's compute each part:

- The derivative of $g(x) = x^3$ is:
$$g'(x) = 3x^2$$
- The derivative of $h(z) = (z-1)(z+1) = z^2 - 1$ is: $h'(z) = 2z$

Chain Rule Application

Now, apply the Chain Rule:

$$\frac{d}{dz}[g(h(z))] = g'(h(z)) \cdot h'(z) = 3(h(z))^2 \cdot 2z = 3(z^2-1)^2 \cdot 2z$$

Simplifying:

$$\frac{d}{dz}[g(h(z))] = 6z(z^2-1)^2$$

Comparison of the Results

Both methods give the same result:

- Direct differentiation: $6z(z^2 - 1)^2$
- Chain Rule application: $6z(z^2 - 1)^2$

Thus, the derivative of the composite function $(g \circ h)(z)$ is:

$$\boxed{6z(z^2 - 1)^2}$$

Example C1: Chain Rule

Problem

- Consider the profit function $\pi(y) = 3y^3 - 5y + 2$ and the production function $y = 4L^{\frac{1}{2}}$.
- The composite profit function is:

$$P(L) = \pi(f(L)) = \pi(4L^{\frac{1}{2}})$$

- Please find the derivative $\frac{d}{dL}P(L)$ using the Chain Rule.

Solution

1) Inside and Outside Functions

- The function $P(L) = \pi(f(L))$ is a composition of two functions:
  - The outside function is $\pi(y) = 3y^3 - 5y + 2$, where $y = f(L)$.
  - The inside function is $f(L) = 4L^{\frac{1}{2}}$.
- To differentiate, we will apply the Chain Rule:

$$\frac{d}{dL}P(L) = \frac{d}{dy}\pi(y) \cdot \frac{d}{dL}f(L)$$

Solution (cont.)

2) Differentiate the Outside Function

- Differentiate $\pi(y) = 3y^3 - 5y + 2$ with respect to y:

$$\frac{d}{dy}\pi(y) = 9y^2 - 5$$

- So, $\frac{d}{dy}\pi(f(L)) = 9\,(f(L))^2 - 5 = 9\left(4L^{\frac{1}{2}}\right)^2 - 5 = 9 \cdot 16L + (-5) = 144L - 5.$

Solution (cont.)

3) Differentiate the Inside Function

- The inside function is $f(L) = 4L^{\frac{1}{2}}$.
- Differentiate it with respect to L:

$$\frac{d}{dL}f(L) = \frac{d}{dL}\left(4L^{\frac{1}{2}}\right) = 4 \times \frac{1}{2}L^{-\frac{1}{2}} = \frac{2}{L^{\frac{1}{2}}}$$

Solution (cont.)

4) Combine the Results

- Now we combine the derivatives:

$$\frac{d}{dL}P(L) = (144L - 5) \cdot \frac{2}{L^{\frac{1}{2}}}$$

- Simplify the expression:

$$\frac{d}{dL}P(L) = \frac{2(144L - 5)}{L^{\frac{1}{2}}}$$

- Final simplified derivative:

$$\frac{d}{dL}P(L) = \frac{288L - 10}{L^{\frac{1}{2}}}$$

Solution (cont.)

Solution

$$\frac{d}{dL}P(L) = \frac{288L - 10}{L^{\frac{1}{2}}}$$

This is the derivative of the composite profit function $P(L) = \pi(f(L))$.

Example C2: Chain Rule

Problem

- Consider the demand function $q(p) = 100 - 2p$ and the cost function $C(q) = 3q^2 + 10q + 5$.
- The composite cost function is:

$$C(p) = C(q(p)) = 3(q(p))^2 + 10q(p) + 5$$

- Please find the derivative $\frac{d}{dp}C(p)$ using the Chain Rule.

Solution

1) Inside and Outside Functions

- The function $C(p) = C(q(p))$ is a composition of two functions:
  - The outside function is $C(q) = 3q^2 + 10q + 5$, where $q = q(p)$.
  - The inside function is $q(p) = 100 - 2p$.
- To differentiate, we will apply the Chain Rule:

$$\frac{d}{dp}C(p) = \frac{d}{dq}C(q) \cdot \frac{d}{dp}q(p)$$

Solution (cont.)

2) Differentiate the Outside Function

- Differentiate $C(q) = 3q^2 + 10q + 5$ with respect to q:

$$\frac{d}{dq}C(q) = 6q + 10$$

- So, $\frac{d}{dq}C(q(p)) = 6q(p) + 10$.
- Substituting $q(p) = 100 - 2p$ into the result:

$$\frac{d}{dq}C(q(p)) = 6(100 - 2p) + 10 = 600 - 12p + 10 = 610 - 12p$$

Solution (cont.)

3) Differentiate the Inside Function

- The inside function is $q(p) = 100 - 2p$.
- Differentiate it with respect to p:

$$\frac{d}{dp}q(p) = \frac{d}{dp}(100 - 2p) = -2$$

Solution (cont.)

4) Combine the Results

- Now we combine the derivatives:

$$\frac{d}{dp}C(p) = (610 - 12p) \cdot (-2)$$

- Simplifying the expression:

$$\frac{d}{dp}C(p) = -2(610 - 12p) = -1220 + 24p$$

Solution (cont.)

Solution

$$\frac{d}{dp}C(p) = 24p - 1220$$

This is the derivative of the composite cost function $C(p) = C(q(p))$.

Example C3: Chain Rule

Problem

- Consider the revenue function $R(p) = p \cdot q(p)$, where $q(p) = 100 - 2p$ is the demand function, and the cost function $C(q) = 4q^2 + 20q + 50$.
- The composite cost function is:

$$C(R(p)) = 4(R(p))^2 + 20R(p) + 50$$

- Please find the derivative $\frac{d}{dp}C(R(p))$ using the Chain Rule.

Solution

1) Inside and Outside Functions

- The function $C(R(p))$ is a composition of two functions:
  - The outside function is $C(R) = 4R^2 + 20R + 50$, where $R = R(p)$.
  - The inside function is $R(p) = p \cdot q(p) = p(100 - 2p)$.
- To differentiate, we will apply the Chain Rule:

$$\frac{d}{dp}C(R(p)) = \frac{d}{dR}C(R) \cdot \frac{d}{dp}R(p)$$

Solution (cont.)

2) Differentiate the Outside Function

- Differentiate $C(R) = 4R^2 + 20R + 50$ with respect to R:

$$\frac{d}{dR}C(R) = 8R + 20$$

- So, $\frac{d}{dR}C(R(p)) = 8R(p) + 20$.
- Substituting $R(p) = p(100 - 2p)$:

$$\frac{d}{dR}C(R(p)) = 8\left(p(100 - 2p)\right) + 20 = 8p(100 - 2p) + 20$$

Solution (cont.)

3) Differentiate the Inside Function

- The inside function is $R(p) = p(100 - 2p)$.
- Differentiate it with respect to p:

$$\frac{d}{dp}R(p) = \frac{d}{dp}\left(p(100 - 2p)\right) = 100 - 4p$$

Solution (cont.)

4) Combine the Results

- Now we combine the derivatives:

$$\frac{d}{dp}C(R(p)) = (8p(100 - 2p) + 20) \cdot (100 - 4p)$$

- This is the derivative of the composite cost function with respect to price p.

Solution(cont.)

Solution

$$\frac{d}{dp}C(R(p)) = (8p(100-2p)+20)\cdot(100-4p)$$

This is the final expression for the derivative of the composite cost function $C(R(p))$.