Problems and Solutions-III

P & S 3

11 October 2024

Example B1

Problem:

Consider the function of

$$f(x) = (x^2 - 1)(x^2 - 5)$$

a) find the critical and inflection points for the function

b) use sign chart where the function increases and decreases

c) use sign chart where the function is concave down and concave up

d) draw the graph of function

e) identify the local maximum and minimum

f) investigate whether the optimal points are global or not

a) calculate the first derivative and find critical points

1)

First, compute the first derivative of the function using the product rule:

$$f'(x) = 4x(x^2 - 3)$$

2)

Critical points occur where $f'(x) = 0$. Solve:

$$4x(x^2 - 3) = 0$$

This gives:

$$x = 0, \quad x = \pm\sqrt{3}$$

Therefore, the critical points are:

$$\boxed{x = 0, \pm\sqrt{3}}$$

a) calculate the second derivative and find the inflection points

3)

Next, compute the second derivative $f''(x)$:

$$f''(x) = 12(x^2 - 1)$$

4)

Inflection points occur where $f''(x) = 0$. Solve:

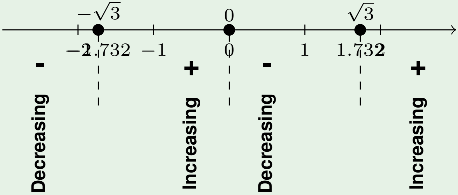
$$12(x^2 - 1) = 0 \quad \Rightarrow \quad x = \pm 1$$

Therefore, the inflection points are:

$$\boxed{x = \pm 1}$$

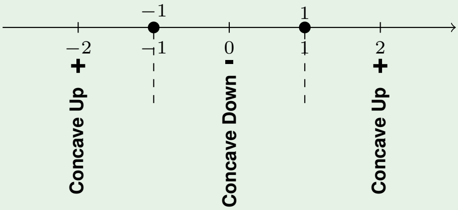
b) use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing



c) use sign chart where the function is concave down and concave up

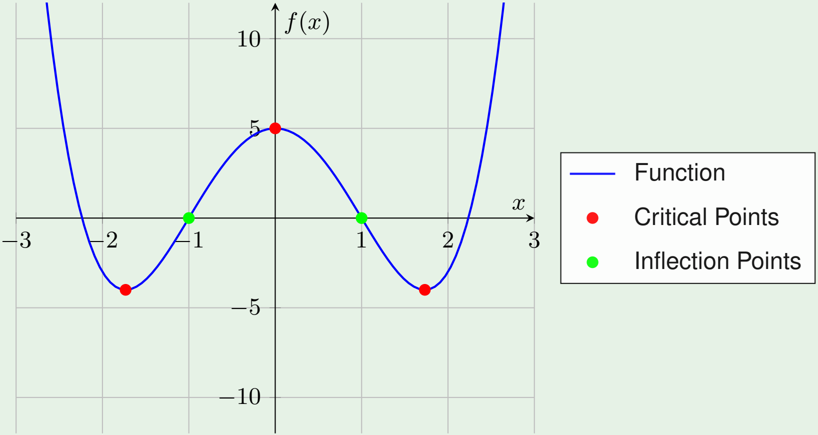
Sign Chart for Concavity



d) draw the graph of the function

Graph of $f(x) = (x^2 - 1)(x^2 - 5)$

$$f(x) = (x^2 - 1)(x^2 - 5)$$



e) identify the local maximum and minimum

Determining Local Maxima and Minima

We have found the critical points at $x = -\sqrt{3}, 0, \sqrt{3}$ by setting $f'(x) = 0$. Now we apply the first derivative test:

- At $x = -\sqrt{3}$: The derivative changes from negative to positive, so $x = -\sqrt{3}$ is a **local minimum**.
- At $x = 0$: The derivative changes from positive to negative, so $x = 0$ is a **local maximum**.
- At $x = \sqrt{3}$: The derivative changes from negative to positive, so $x = \sqrt{3}$ is a **local minimum**.

The function values at these critical points are:

$$f(0) = 5, \quad f(\pm\sqrt{3}) = -4$$

Therefore, the local maxima and minima are:

Local maximum at $x = 0, f(0) = 5$

Local minima at $x = \pm\sqrt{3}, f(\pm\sqrt{3}) = -4$

f) investigate whether the optimal points are global or not

Global Extrema of $f(x) = (x^2 - 1)(x^2 - 5)$

We already know that the function has local extrema at the critical points:

- Local maximum at $x = 0$, $f(0) = 5$.
- Local minima at $x = \pm\sqrt{3}$, $f(\pm\sqrt{3}) = -4$.

Now, we analyze the end behavior of the function:

$$f(x) = x^4 - 6x^2 + 5$$

As $x \to \infty$ or $x \to -\infty$, the leading term x^4 dominates, so:

$$f(x) \to \infty \quad \text{as} \quad x \to \pm\infty$$

Conclusion:

- The function has no global maximum because $f(x) \to \infty$ as $x \to \pm\infty$.
- The function has a global minimum at $x = -\sqrt{3}$ and $x = \sqrt{3}$, where $f(x) = -4$.

Example B2

Problem:

Consider the function

$$f(x) = x^3 - 6x^2 + 9x + 1$$

a) Find the critical and inflection points for the function.

b) Use a sign chart to determine where the function is increasing and decreasing.

c) Use a sign chart to determine where the function is concave down and concave up.

d) Sketch the graph of the function showing the critical and inflection points.

e) Identify the local maximum and minimum.

f) Investigate whether the optimal points are global or not.

a) Calculate the first derivative and find critical points

1)

First, compute the first derivative of the function:

$$f'(x) = 3x^2 - 12x + 9$$

2)

Critical points occur where $f'(x) = 0$. Solve:

$$3x^2 - 12x + 9 = 0$$

This factors as:

$$3(x^2 - 4x + 3) = 0 \quad \Rightarrow \quad (x-1)(x-3) = 0$$

Therefore, the critical points are:

$$\boxed{x = 1, \; x = 3}$$

a) Calculate the second derivative and find inflection points

3)

Next, compute the second derivative $f''(x)$:

$$f''(x) = 6x - 12$$

4)

Inflection points occur where $f''(x) = 0$. Solve:

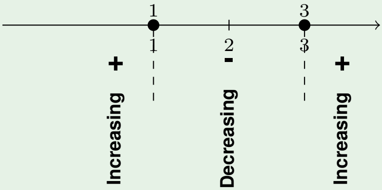
$$6x - 12 = 0 \quad \Rightarrow \quad x = 2$$

Therefore, the inflection point is:

$$\boxed{x = 2}$$

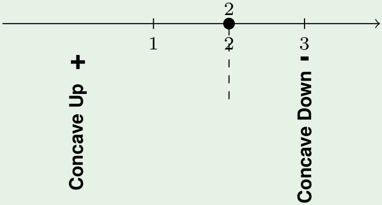
b) Use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing



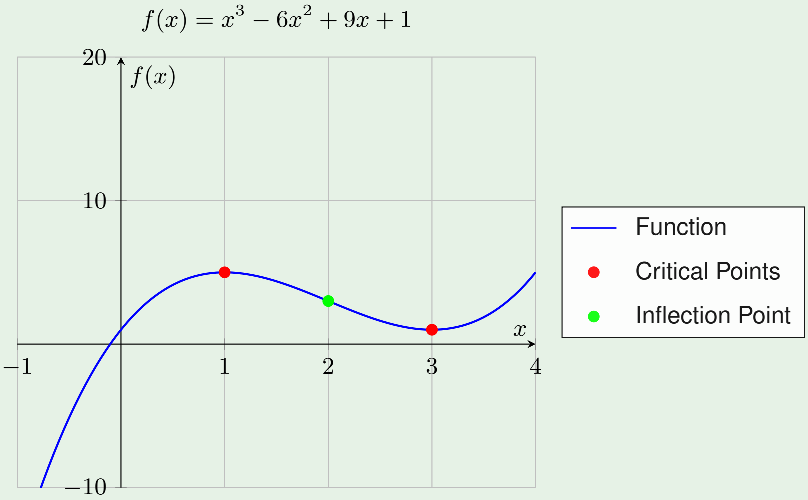
c) Use sign chart where the function is concave down and concave up

Sign Chart for Concavity



d) Sketch the graph of the function

Graph of $f(x) = x^3 - 6x^2 + 9x + 1$



e) Identify the local maximum and minimum

Local Maximum and Minimum

We have found the critical points at $x = 1$ and $x = 3$. Now, we apply the first derivative test:

- At $x = 1$: The function changes from increasing to decreasing, so $x = 1$ is a **local maximum**.
- At $x = 3$: The function changes from decreasing to increasing, so $x = 3$ is a **local minimum**.

The function values at these critical points are:

$$f(1) = 1, \quad f(3) = 1$$

Therefore, the local maximum and minimum are:

Local maximum at $x = 1, f(1) = 1$

Local minimum at $x = 3, f(3) = 1$

f) Investigate Global Extrema

Global Extrema of $f(x) = x^3 - 6x^2 + 9x + 1$

We already know:

- The function has a local maximum at $x = 1$, with $f(1) = 5$.
- The function has a local minimum at $x = 3$, with $f(3) = 1$.

- As $x \to \infty$,
$$f(x) \to \infty \quad \text{as} \quad x \to \infty.$$
- As $x \to -\infty$,
$$f(x) \to -\infty \quad \text{as} \quad x \to -\infty.$$

Conclusion:

- There is no global maximum because the function increases without bound as $x \to \infty$.
- There is no global minimum because the function decreases without bound as $x \to -\infty$.

Example 3.5

Problem:

Consider the function

$$f(x) = x^4 - 4x^3 + 4x^2 + 4$$

a) Find the critical and inflection points for the function

b) Use a sign chart to determine where the function increases and decreases

c) Use a sign chart to determine where the function is concave up and concave down

d) Draw the graph of the function

e) Identify the local maximum and minimum

f) Investigate whether the optimal points are global or not

a) Calculate the first derivative and find critical points

1) First Derivative

Compute the first derivative of the function:

$$f'(x) = 4x^3 - 12x^2 + 8x$$

2) Critical Points

Critical points occur when $f'(x) = 0$. Solve the equation:

$$4x(x^2 - 3x + 2) = 0$$

Factor the quadratic:

$$4x(x-1)(x-2) = 0$$

Therefore, the critical points are:

$$\boxed{x = 0, 1, 2}$$

a) Calculate the second derivative and find inflection points

3) Second Derivative

Compute the second derivative of the function:

$$f''(x) = 12x^2 - 24x + 8$$

4) Inflection Points

Inflection points occur when $f''(x) = 0$. Solve the equation:

$$12x^2 - 24x + 8 = 0$$

Divide by 4:

$$3x^2 - 6x + 2 = 0$$

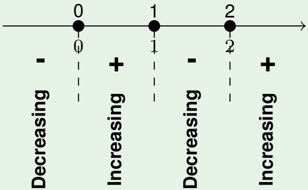
Using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6}$$

$$\boxed{x = 1 + \frac{\sqrt{3}}{3}, \quad x = 1 - \frac{\sqrt{3}}{3}}$$

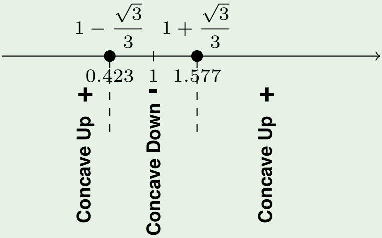
b) Use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing



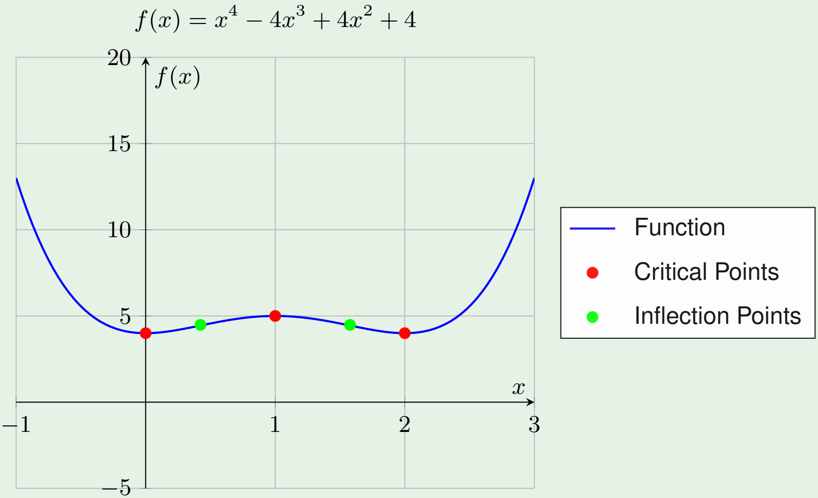
c) Use sign chart where the function is concave down and concave up

Sign Chart for Concavity



d) Sketch the graph of the function

Graph of $f(x) = x^4 - 4x^3 + 4x^2 + 4$



e) Local Maxima and Minima

1) Apply the Second Derivative Test

To determine whether the critical points correspond to local maxima or minima, we apply the second derivative test:

$$f''(x) = 12x^2 - 24x + 8$$

At $x = 0$:

$$f''(0) = 12(0)^2 - 24(0) + 8 = 8 \quad \text{(positive, so local minimum at } x = 0\text{)}$$

At $x = 1$:

$$f''(1) = 12(1)^2 - 24(1) + 8 = -4 \quad \text{(negative, so local maximum at } x = 1\text{)}$$

At $x = 2$:

$$f''(2) = 12(2)^2 - 24(2) + 8 = 8 \quad \text{(positive, so local minimum at } x = 2\text{)}$$

If $f''(x) > 0$, it's a local minimum. If $f''(x) < 0$, it's a local maximum. If $f''(x) = 0$, the test is inconclusive.

f) Investigate whether the optimal points are global or not

Global Minima or Maxima

Note that $x = 0$ and $x = 2$ are global minimum of f. However, $x = 1$ is not definitely a global maximum, since f eventually takes on arbitrarily large values as $x \to \infty$

The function has global minimum at:

- **Global minimum at** $x = 0$ and $x = 2$ with $f(0) = f(2) = 4$

$f(x) \to \infty$ as $x \to \infty$ and $x \to -\infty$, the function has no global maximum.

The function increases without bound as $x \to \infty$ or $x \to -\infty$, so there are no global maximum.

Example B3

Problem:

Consider the function

$$f(x) = x^3 - 3x^2 + 2x + 5$$

a) Find the critical points and inflection points of the function

b) Use a sign chart to determine where the function is increasing and decreasing

c) Use a sign chart to determine where the function is concave up and concave down

d) Sketch the graph of the function

e) Identify the local maximum and minimum

f) Investigate whether the optimal points are global or not

a) Calculate the first derivative and find critical points

1) First Derivative

Compute the first derivative of the function:

$$f'(x) = 3x^2 - 6x + 2$$

2) Critical Points

Critical points occur when $f'(x) = 0$. Solve the equation:

$$3x^2 - 6x + 2 = 0$$

Divide by 3:

$$x^2 - 2x + \frac{2}{3} = 0$$

Using the quadratic formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(\frac{2}{3})}}{2(1)} = \frac{2 \pm \sqrt{4 - \frac{8}{3}}}{2}$$

a) Calculate the first derivative and find critical points

2) Critical Points

$$x = \frac{2 \pm \sqrt{\frac{12}{3} - \frac{8}{3}}}{2} = \frac{2 \pm \sqrt{\frac{4}{3}}}{2}$$

$$x = 1 \pm \frac{\sqrt{3}}{3}$$

Therefore, the critical points are:

$$\boxed{x = 1 + \frac{\sqrt{3}}{3}, \quad x = 1 - \frac{\sqrt{3}}{3}}$$

b) Calculate the second derivative and find inflection points

3) Second Derivative

Compute the second derivative of the function:

$$f''(x) = 6x - 6$$

4) Inflection Points

Inflection points occur when $f''(x) = 0$. Solve the equation:

$$6x - 6 = 0$$

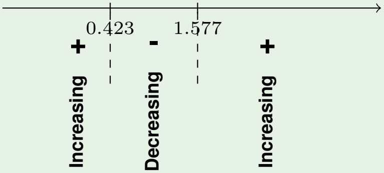
$$x = 1$$

Therefore, the inflection point is:

$$\boxed{x = 1}$$

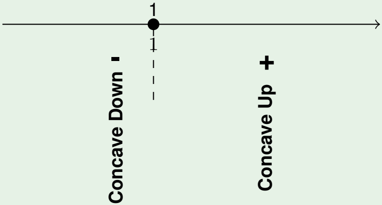
c) Use sign chart where the function is increasing and decreasing

Sign Chart for Increasing/Decreasing



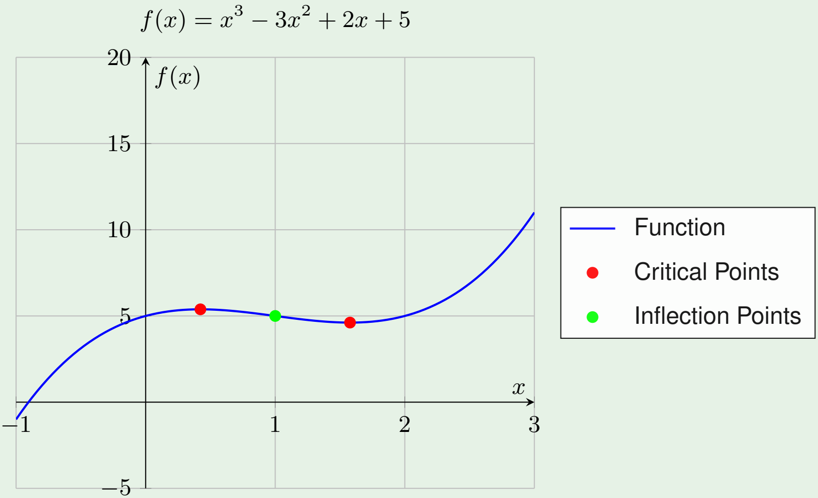
d) Use sign chart where the function is concave up and concave down

Sign Chart for Concavity



e) Sketch the graph of the function

Graph of $f(x) = x^3 - 3x^2 + 2x + 5$



f) Local Maxima and Minima

Apply the Second Order Condition

To determine whether the critical points correspond to local maxima or minima, we apply the second derivative test:

$$f''(x) = 6x - 6$$

At $x = 1 + \frac{\sqrt{3}}{3}$:

$$f''(1 + \frac{\sqrt{3}}{3}) = 6\left(1 + \frac{\sqrt{3}}{3}\right) - 6 = \text{positive value, so local minimum.}$$

At $x = 1 - \frac{\sqrt{3}}{3}$:

$$f''(1 - \frac{\sqrt{3}}{3}) = 6\left(1 - \frac{\sqrt{3}}{3}\right) - 6 = \text{negative value, so local maximum.}$$

g) Investigate whether the optimal points are global or not

Global Extrema

Any strictly increasing or strictly decreasing function whose domain is an *open interval* will not have a maximum or a minimum in its domain

- As $x \to +\infty$, $f(x) \to +\infty$.
- As $x \to -\infty$, $f(x) \to -\infty$.

Conclusion:

- The function does **not have a global maximum** because as $x \to +\infty$, the function increases without bound.
- The function does **not have a global minimum** because as $x \to -\infty$, the function decreases without bound.

Therefore, while the function has a local maximum and a local minimum, it does not have any global extrema.

Example 3.6

Problem:

The membership of the Association of Smart Statisticians is given by the function

$$f(x) = 2x^3 - 45x^2 + 300x + 500$$

where x is the number of years after 1960. Find the largest and smallest membership between 1960 and 1980, i.e., for $x \in [0, 20]$, and determine when these extreme values occur. Mathematically, this is the problem of maximizing

f(x)=2x$^3 - 45x^2 + 300x + 500 forxintheclosedinterval[0, 20]$.

First Derivative and Critical Points

First Derivative

To find the critical points, we first compute the first derivative of the function:

$$f'(x) = 6x^2 - 90x + 300$$

Critical Points

Critical points occur where $f'(x) = 0$. Solve the equation:

$$6x^2 - 90x + 300 = 0$$

Dividing through by 6:

$$x^2 - 15x + 50 = 0$$

Using the quadratic formula:

$$x = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(1)(50)}}{2(1)} = \frac{15 \pm \sqrt{225 - 200}}{2} = \frac{15 \pm \sqrt{25}}{2} = \frac{15 \pm 5}{2}$$

The solutions are:

$$x = 10 \quad \text{or} \quad x = 5$$

Evaluate the Function

Evaluate at Critical Points and Endpoints

Next, we evaluate the function $f(x)$ at the critical points and the endpoints of the interval $[0, 20]$:

$$f(0) = 2(0)^3 - 45(0)^2 + 300(0) + 500 = 500$$

$$f(5) = 2(5)^3 - 45(5)^2 + 300(5) + 500 = 1125$$

$$f(10) = 2(10)^3 - 45(10)^2 + 300(10) + 500 = 0$$

$$f(20) = 2(20)^3 - 45(20)^2 + 300(20) + 500 = 500$$

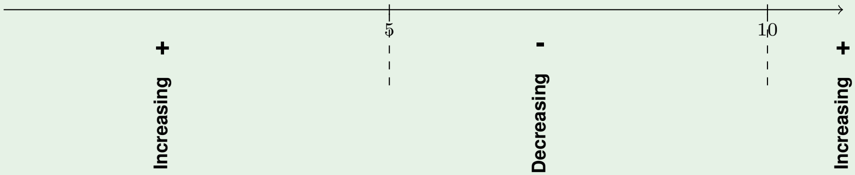
Sign Chart for Increasing/Decreasing

Sign Chart for Increasing/Decreasing

To determine where the function is increasing or decreasing, we use the first derivative:

$$f'(x) = 6x^2 - 90x + 300$$

Critical points are $x = 5$ and $x = 10$. We now test the sign of $f'(x)$ in the intervals $(-\infty, 5)$, $(5, 10)$, and $(10, \infty)$.



Second Derivative and Concavity

Second Derivative

To determine where the function is concave up or concave down, we compute the second derivative:

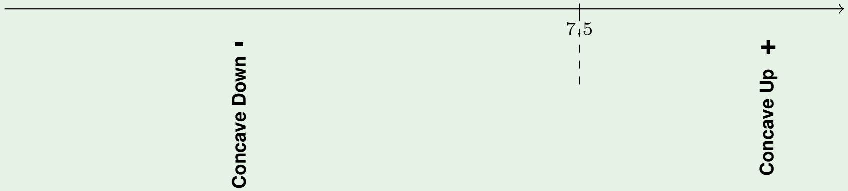
$$f''(x) = 12x - 90$$

Setting $f''(x) = 0$, we find the inflection point:

$$12x - 90 = 0 \quad \Rightarrow \quad x = 7.5$$

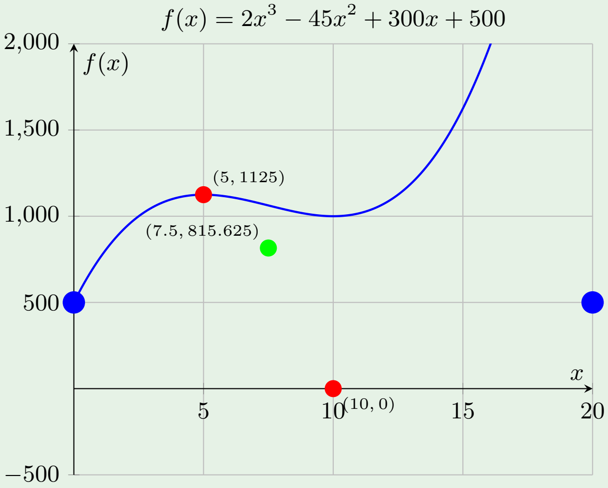
Sign Chart for Concavity

We now analyze the concavity by testing the sign of $f''(x)$ in the intervals $(-\infty, 7.5)$ and $(7.5, \infty)$.



Graph of the Function

Graph of $f(x) = 2x^3 - 45x^2 + 300x + 500$



Global Maximum and Minimum

Global Maximum and Minimum

The function values at the critical points and endpoints are:

$$f(0) = 500, \quad f(5) = 1125, \quad f(10) = 0, \quad f(20) = 500$$

- The global maximum occurs at $x = 5$ with $f(5) = 1125$. - The global minimum occurs at $x = 10$ with $f(10) = 0$.