# Problems and Solutions-II

P & S 2

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# Example 2.22

## Problem

Suppose that the total cost of manufacturing $x$ units of a certain commodity is $C(x) = 2x^2 + 6x + 12$. Use differentials to approximate the cost of producing the 21st unit. Compare this estimate with the actual cost of producing the 21st unit.

# Solution

## 1) Find the derivative of $C(x)$

The total cost function is given as:

$$C(x) = 2x^2 + 6x + 12$$

The derivative of $C(x)$, denoted as $C'(x)$, is:

$$C'(x) = \frac{d}{dx}(2x^2 + 6x + 12) = 4x + 6$$

## 2) Approximate the cost of producing the 21st unit using differentials

The differential $dC$ represents the approximate change in cost when $x$ increases by $dx$:

$$dC \approx C'(x) \cdot dx$$

We want to approximate the cost of producing the 21st unit, so we calculate $dC$ at $x = 20$ and $dx = 1$:

$$dC \approx C'(20) \cdot 1 = 4(20) + 6 = 86$$

# Solution (Cont.)

## 3) Actual cost of producing the 21st unit

To find the actual cost of producing the 21st unit, we calculate the difference in cost between producing 21 units and producing 20 units:

$$C(21) = 2(21)^2 + 6(21) + 12 = 1020$$

$$C(20) = 2(20)^2 + 6(20) + 12 = 932$$

The actual cost of producing the 21st unit is:

$$\text{Actual cost of the 21st unit} = C(21) - C(20) = 1020 - 932 = 88$$

## Comparison

- The estimated cost using differentials is 86.
- The actual cost is 88.

# Example 2.23

## Problem

A manufacturer's total cost is given by the function:

$$C(x) = 0.1x^3 - 0.25x^2 + 300x + 100 \quad \text{(in dollars)},$$

where $x$ is the level of production. Estimate the effect on the total cost of an increase in the level of production from 6 to 6.1 units.

# Solution

## 1) Find the derivative of $C(x)$

The total cost function is given as:

$$C(x) = 0.1x^3 - 0.25x^2 + 300x + 100$$

The derivative of $C(x)$, denoted as $C'(x)$, is:

$$C'(x) = \frac{d}{dx}(0.1x^3 - 0.25x^2 + 300x + 100)$$

Using the power rule of differentiation:

$$C'(x) = 0.3x^2 - 0.5x + 300$$

# Solution (Cont.)

## 2) Approximate the change in cost

We want to estimate the effect on the total cost of an increase in production from 6 to 6.1 units.
We use the derivative $C'(x)$ to estimate the rate of change of the cost at $x = 6$. The change in $x$ is $\Delta x = 6.1 - 6 = 0.1$.
Now, we evaluate the derivative at $x = 6$:

$$C'(6) = 0.3(6)^2 - 0.5(6) + 300 = 0.3(36) - 3 + 300 = 10.8 - 3 + 300 = 307.8$$

Thus, the rate of change of cost at $x = 6$ is approximately 307.8 dollars per unit.
To estimate the change in cost, we multiply this rate by the change in production $\Delta x$:

$$\Delta C \approx C'(6) \cdot \Delta x = 307.8 \cdot 0.1 = 30.78$$

The estimated increase in the total cost due to the increase in production from 6 to 6.1 units is approximately $30.78.

# Example 2.24

## Problem

It is estimated that $t$ years from now, the population of a certain town will be

$$F(t) = 40 - \frac{8}{t+2}.$$

Use differentials to estimate the amount by which the population will increase during the next six months.

# Solution

## 1) Differentiate the Function

We are given the function for the population:

$$F(t) = 40 - \frac{8}{t+2}.$$

We differentiate $F(t)$ with respect to $t$:

$$F'(t) = \frac{d}{dt}(40) - \frac{d}{dt}\left(\frac{8}{t+2}\right) = 0 - \frac{-8}{(t+2)^2} = \frac{8}{(t+2)^2}.$$

## 2) Differential Approximation

Using the differential approximation:

$$dF \approx F'(t)\,dt = \frac{8}{(t+2)^2} \cdot 0.5.$$

# Solution (cont.)

## 3) Evaluate at $t = 0$

At $t = 0$, we have:

$$dF \approx \frac{8}{(0+2)^2} \cdot 0.5 = \frac{8}{4} \cdot 0.5 = 1.$$

Thus, the population will increase by approximately 1 person over the next six months.

# Example A1

## Problem

Suppose that the total cost of manufacturing $x$ units of a certain commodity is $C(x) = 3x^2 + 8x + 15$. Use differentials to approximate the cost of producing the 30th unit. Compare this estimate with the actual cost of producing the 30th unit.

# Solution

## 1) Find the derivative of $C(x)$

The total cost function is given as:

$$C(x) = 3x^2 + 8x + 15$$

The derivative of $C(x)$, denoted as $C'(x)$, is:

$$C'(x) = \frac{d}{dx}(3x^2 + 8x + 15) = 6x + 8$$

## 2) Approximate the cost of producing the 30th unit using differentials

The differential $dC$ represents the approximate change in cost when $x$ increases by $dx$:

$$dC \approx C'(x) \cdot dx$$

We want to approximate the cost of producing the 30th unit, so we calculate $dC$ at $x = 29$ and $dx = 1$:

$$dC \approx C'(29) \cdot 1 = 6(29) + 8 = 174 + 8 = 182$$

# Solution (Cont.)

## 3) Actual cost of producing the 30th unit

To find the actual cost of producing the 30th unit, we calculate the difference in cost between producing 30 units and producing 29 units:

$$C(30) = 3(30)^2 + 8(30) + 15 = 2700 + 240 + 15 = 2955$$

$$C(29) = 3(29)^2 + 8(29) + 15 = 2523 + 232 + 15 = 2770$$

The actual cost of producing the 30th unit is:

$$\text{Actual cost of the 30th unit} = C(30) - C(29) = 2955 - 2770 = 185$$

## Comparison

- The estimated cost using differentials is 182.
- The actual cost is 185.

# Example A2

## Problem

A company's total cost is given by the function:

$$C(x) = 0.05x^4 - 0.4x^3 + 250x^2 + 500x + 200 \quad \text{(in dollars)},$$

where $x$ represents the number of units produced. Estimate the effect on the total cost of increasing the production level from 8 to 8.2 units.

# Solution

## 1) Find the derivative of $C(x)$

The total cost function is given as:

$$C(x) = 0.05x^4 - 0.4x^3 + 250x^2 + 500x + 200$$

The derivative of $C(x)$, denoted as $C'(x)$, is:

$$C'(x) = \frac{d}{dx}(0.05x^4 - 0.4x^3 + 250x^2 + 500x + 200)$$

Using the power rule of differentiation:

$$C'(x) = 0.2x^3 - 1.2x^2 + 500x + 500$$

# Solution (Cont.)

## 2) Approximate the change in cost

We want to estimate the effect on the total cost of an increase in production from 8 to 8.2 units.
The change in $x$ is $\Delta x = 8.2 - 8 = 0.2$.
Now, we evaluate the derivative at $x = 8$:

$$C'(8) = 0.2(8)^3 - 1.2(8)^2 + 500(8) + 500 =$$

$$0.2(512) - 1.2(64) + 4000 + 500 = 102.4 - 76.8 + 4000 + 500 = 4525.6$$

Thus, the rate of change of cost at $x = 8$ is approximately 4525.6 dollars per unit.
To estimate the change in cost, we multiply this rate by the change in production $\Delta x$:

$$\Delta C \approx C'(8) \cdot \Delta x = 4525.6 \cdot 0.2 = 905.12$$

The estimated increase in the total cost due to the increase in production from 8 to 8.2 units is approximately $905.12.

# Example A3

## Problem

The temperature in a certain city at $t$ hours from now is estimated by the function

$$T(t) = 30 + \frac{10}{t+3}.$$

Use differentials to estimate the change in temperature over the next 4 hours.

# Solution

## 1) Differentiate the Function

We are given the function for the temperature:

$$T(t) = 30 + \frac{10}{t+3}.$$

We differentiate $T(t)$ with respect to $t$:

$$T'(t) = \frac{d}{dt}(30) - \frac{d}{dt}\left(\frac{10}{t+3}\right) = 0 + \frac{-10}{(t+3)^2} = -\frac{10}{(t+3)^2}.$$

## 2) Differential Approximation

Using the differential approximation:

$$dT \approx T'(t)\,dt = -\frac{10}{(t+3)^2} \cdot 4.$$

# Solution (cont.)

## 3) Evaluate at $t = 0$

At $t = 0$, we have:

$$dT \approx -\frac{10}{(0+3)^2} \cdot 4 = -\frac{10}{9} \cdot 4 = -\frac{40}{9} \approx -4.44.$$

Thus, the temperature will decrease by approximately 4.44 degrees over the next 4 hours.