Problems and Solutions-I

P & S 1

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Example 2.8 (a)

Find the formula for the linear function whose graph has slope 2 and y-intercept (0,3).
Solution: The formula for a linear function is given by the equation:

$$y = mx + b$$

where m is the slope and b is the y-intercept.
Given that the slope is 2 and the y-intercept is 3, the equation of the line is:

$$y = 2x + 3$$

Example 2.8 (b)

Find the formula for the linear function whose graph has slope -3 and y-intercept (0,0).
Solution: The formula for a linear function is given by the equation:

$$y = mx + b$$

where m is the slope and b is the y-intercept.
Given that the slope is -3 and the y-intercept is $(0, 0)$, the equation of the line is:

$$y = -3x + 0$$

Simplifying the equation:

$$y = -3x$$

Example 2.8 (c)

Find the formula for the linear function whose graph has slope 4 and goes through the point (1,1).
Solution: The formula for a linear function is given by the equation:

$$y = mx + b$$

where m is the slope and b is the y-intercept.
Given that the slope is 4 and the y-intercept is $(1, 1)$, we need to find the equation of the line. However, the y-intercept value b is not 0 here, it's the point where the graph intersects the y-axis. For this case, the point $(1, 1)$ represents a specific point on the line rather than just the y-intercept.
To write the equation, we'll use the point-slope form of a line:

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1) = (1, 1)$ is a point on the line and $m = 4$ is the slope. Substituting these values into the point-slope form:

$$y - 1 = 4(x - 1)$$

Simplifying the equation:

$$y - 1 = 4x - 4 \quad \text{and} \quad y = 4x - 3$$

Example 2.8 (d)

Find the formula for the linear function whose graph has slope -2 and goes through the point (2,-2).
Solution: The formula for a linear function is given by the equation:

$$y = mx + b$$

where m is the slope and b is the y-intercept.
Given that the slope is -2 and the line goes through the point $(2, -2)$, we can use the point-slope form of the equation:

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1) = (2, -2)$ and $m = -2$. Substituting these values into the point-slope form:

$$y - (-2) = -2(x - 2)$$

Simplifying the equation:

$$y + 2 = -2(x - 2) \quad \text{and} \quad y + 2 = -2x + 4 \quad \text{so} \quad y = -2x + 2$$

Example 2.8 (e)

Find the formula for the linear function that goes through the points $(2,3)$ and $(4,5)$.
Solution: The formula for a linear function is given by the equation:

$$y = mx + b$$

where m is the slope and b is the y-intercept.
To find the slope, we use the formula for the slope between two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $(x_1, y_1) = (2,3)$ and $(x_2, y_2) = (4,5)$.
Substituting these values into the slope formula:

$$m = \frac{5-3}{4-2} = \frac{2}{2} = 1$$

Now that we know the slope $m = 1$, we can use the point-slope form of the equation:

$$y - 3 = 1(x - 2)$$

Simplifying this equation:

$$y - 3 = x - 2 \quad \text{so} \quad y = x + 1$$

Example 2.8 (f)

Find the formula for the linear function that goes through the points $(2, -4)$ and $(0, 3)$.
Solution:
To find the slope, we use the formula for the slope between two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $(x_1, y_1) = (2, -4)$ and $(x_2, y_2) = (0, 3)$.
Substituting these values into the slope formula:

$$m = \frac{3 - (-4)}{0 - 2} = \frac{3 + 4}{-2} = \frac{7}{-2} = -\frac{7}{2}$$

Now that we know the slope $m = -\frac{7}{2}$, we can use the point-slope form of the equation:

$$y - (-4) = -\frac{7}{2}(x - 2)$$

Simplifying this equation:

$$y + 4 = -\frac{7}{2}(x - 2) \quad \text{so} \quad y = -\frac{7}{2}x + 3$$

Example 2.11 (a)

Find the derivative of $f(x) = -7x^3$ at an arbitrary point.
Solution: The given function is:

$$f(x) = -7x^3$$

To find the derivative, we apply the power rule:

$$f'(x) = \frac{d}{dx}\left(-7x^3\right)$$

The power rule states that $\frac{d}{dx}(ax^n) = a \cdot n \cdot x^{n-1}$, where $a = -7$ and $n = 3$. Therefore:

$$f'(x) = -7 \cdot 3 \cdot x^{3-1} = -21x^2$$

Thus, the derivative is:

$$f'(x) = -21x^2$$

Example 2.11 (b)

Find the derivative of $f(x) = 12x^{-2}$ at an arbitrary point.
Solution: The given function is:

$$f(x) = 12x^{-2}$$

To find the derivative, we apply the power rule:

$$f'(x) = \frac{d}{dx}\left(12x^{-2}\right)$$

The power rule states that $\frac{d}{dx}(ax^n) = a \cdot n \cdot x^{n-1}$, where $a = 12$ and $n = -2$.
Therefore:

$$f'(x) = 12 \cdot (-2) \cdot x^{-2-1} = -24x^{-3}$$

Thus, the derivative is:

$$f'(x) = -24x^{-3}$$

Example 2.11 (c)

Find the derivative of $f(x) = 3x^{-3/2}$ at an arbitrary point.
Solution: The given function is:

$$f(x) = 3x^{-3/2}$$

To find the derivative, we apply the power rule:

$$f'(x) = \frac{d}{dx}\left(3x^{-3/2}\right)$$

The power rule states that $\frac{d}{dx}(ax^n) = a \cdot n \cdot x^{n-1}$, where $a = 3$ and $n = -3/2$.
Therefore:

$$f'(x) = 3 \cdot \left(-\frac{3}{2}\right) \cdot x^{-3/2-1} = -9/2x^{-5/2}$$

Thus, the derivative is:

$$f'(x) = -9/2x^{-5/2}$$

Example 2.11 (d)

Find the derivative of $f(x) = \frac{1}{2}\sqrt{x}$ at an arbitrary point.

Solution: The given function is:

$$f(x) = \frac{1}{2}\sqrt{x} = \frac{1}{2}x^{\frac{1}{2}}$$

To find the derivative, we apply the power rule:

$$f'(x) = \frac{d}{dx}\left(\frac{1}{2}x^{\frac{1}{2}}\right)$$

The power rule states that $\frac{d}{dx}(ax^n) = a \cdot n \cdot x^{n-1}$, where $a = \frac{1}{2}$ and $n = \frac{1}{2}$. Therefore:

$$f'(x) = \frac{1}{2} \cdot \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{4}x^{-\frac{1}{2}}$$

Thus, the derivative is:

$$f'(x) = \frac{1}{4}x^{-\frac{1}{2}} = \frac{1}{4\sqrt{x}}$$

Example 2.11 (e)

Find the derivative of $f(x) = 3x^2 - 9x + 7x^{\frac{2}{5}} - 3x^{\frac{1}{2}}$ at an arbitrary point.
Solution: The given function is:

$$f(x) = 3x^2 - 9x + 7x^{\frac{2}{5}} - 3x^{\frac{1}{2}}$$

- For the first term $3x^2$:

$$\frac{d}{dx}\left(3x^2\right) = 6x$$

- For the second term $-9x$:

$$\frac{d}{dx}(-9x) = -9$$

- For the third term $+7x^{\frac{2}{5}}$:

$$\frac{d}{dx}\left(+7x^{\frac{2}{5}}\right) = +\frac{14}{5}x^{-\frac{3}{5}}$$

- For the fourth term $-3x^{\frac{1}{2}}$:

$$\frac{d}{dx}\left(-3x^{\frac{1}{2}}\right) = -\frac{3}{2}x^{-\frac{1}{2}}$$

Combining all the derivatives:

$$f'(x) = 6x - 9 + \frac{14}{5}x^{-\frac{3}{5}} - \frac{3}{2}x^{-\frac{1}{2}}$$

Example 2.11 (f)

Find the derivative of $f(x) = 4x^5 - 3x^{\frac{1}{2}}$ at an arbitrary point.
Solution: The given function is:

$$f(x) = 4x^5 - 3x^{\frac{1}{2}}$$

We differentiate each term using the power rule:
- For the first term $4x^5$:

$$\frac{d}{dx}\left(4x^5\right) = 20x^4$$

- For the second term $-3x^{\frac{1}{2}}$:

$$\frac{d}{dx}\left(-3x^{\frac{1}{2}}\right) = -\frac{3}{2}x^{-\frac{1}{2}}$$

Combining the results:

$$f'(x) = 20x^4 - \frac{3}{2}x^{-\frac{1}{2}}$$

Example 2.11 (g)

Find the derivative of $f(x) = (x^2 + 1)(x^2 + 3x + 2)$ at an arbitrary point.
Solution:
We use the product rule for differentiation:

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

where $u(x) = x^2 + 1$ and $v(x) = x^2 + 3x + 2$.
First, differentiate $u(x)$ and $v(x)$:

$$u'(x) = 2x \quad \text{and} \quad v'(x) = 2x + 3$$

Solution (continued)

Now, applying the product rule:

$$f'(x) = (2x)(x^2 + 3x + 2) + (x^2 + 1)(2x + 3)$$

Expanding each part:

$$f'(x) = (2x^3 + 6x^2 + 4x) + (2x^3 + 3x^2 + 2x + 3)$$

Combine like terms:

$$f'(x) = 4x^3 + 9x^2 + 6x + 3$$

Example 2.11 (h)

Find the derivative of $f(x) = \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(4x^5 - 3\sqrt{x}\right)$ at an arbitrary point.

Solution:

We use the product rule for differentiation:

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

where $u(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ and $v(x) = 4x^5 - 3\sqrt{x} = 4x^5 - 3x^{\frac{1}{2}}$.
First, differentiate $u(x)$ and $v(x)$:

$$u'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \quad \text{and} \quad v'(x) = 20x^4 - \frac{3}{2}x^{-\frac{1}{2}}$$

Solution (continued)

Now, applying the product rule:

$$f'(x) = \left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\right) \cdot (4x^5 - 3x^{\frac{1}{2}}) + \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) \cdot \left(20x^4 - \frac{3}{2}x^{-\frac{1}{2}}\right)$$

Expanding each term:

$$f'(x) = \left(\frac{1}{2}x^{-\frac{1}{2}} \cdot (4x^5 - 3x^{\frac{1}{2}})\right) - \left(\frac{1}{2}x^{-\frac{3}{2}} \cdot (4x^5 - 3x^{\frac{1}{2}})\right)$$

+

$$\left(x^{\frac{1}{2}} \cdot (20x^4 - \frac{3}{2}x^{-\frac{1}{2}})\right) + \left(x^{-\frac{1}{2}} \cdot (20x^4 - \frac{3}{2}x^{-\frac{1}{2}})\right)$$

Example 2.11 (i)

Find the derivative of $f(x) = \dfrac{x-1}{x+1}$ at an arbitrary point.

Solution:

We use the quotient rule for differentiation:

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

where $u(x) = x - 1$ and $v(x) = x + 1$.
First, differentiate $u(x)$ and $v(x)$:

$$u'(x) = 1 \quad \text{and} \quad v'(x) = 1$$

Now, applying the quotient rule:

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

Simplifying:

$$f'(x) = \frac{(x+1) - (x-1)}{(x+1)^2}$$

$$f'(x) = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Example 2.11 (j)

Find the derivative of $f(x) = \dfrac{x}{x^2+1}$ at an arbitrary point.

Solution:

We use the quotient rule for differentiation:

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

where $u(x) = x$ and $v(x) = x^2 + 1$.

First, differentiate $u(x)$ and $v(x)$:

$$u'(x) = 1 \quad \text{and} \quad v'(x) = 2x$$

Now, applying the quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2}$$

Simplifying:

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} \quad \text{and} \quad f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

Example 2.11 (k)

Find the derivative of $f(x) = \left(x^5 - 3x^2\right)^7$ at an arbitrary point.
Solution:
We apply the chain rule for differentiation:

$$f'(x) = g'(h(x)) \cdot h'(x)$$

where $g(u) = u^7$ and $h(x) = x^5 - 3x^2$.
First, differentiate $g(u) = u^7$ and $h(x) = x^5 - 3x^2$:

$$g'(u) = 7u^6 \quad \text{and} \quad h'(x) = 5x^4 - 6x$$

Now, apply the chain rule:

$$f'(x) = 7\left(x^5 - 3x^2\right)^6 \cdot (5x^4 - 6x)$$

Example 2.11 (I)

Find the derivative of $f(x) = 5\left(x^5 - 6x^2 + 3x\right)^{\frac{2}{3}}$ at an arbitrary point.
Solution:
We apply the constant multiple rule and the chain rule:

$$f'(x) = 5 \cdot \frac{2}{3}\left(x^5 - 6x^2 + 3x\right)^{\frac{2}{3}-1} \cdot (5x^4 - 12x + 3)$$

Simplify the exponent:

$$f'(x) = \frac{10}{3}\left(x^5 - 6x^2 + 3x\right)^{-\frac{1}{3}} \cdot (5x^4 - 12x + 3)$$

Example 2.11 (m)

Find the derivative of $f(x) = \left(x^3 + 2x\right)^3 (4x + 5)^2$ at an arbitrary point.
Solution:
Since the function is a product of two terms, we apply the product rule:

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

where:

$$u(x) = \left(x^3 + 2x\right)^3, \quad v(x) = (4x + 5)^2$$

Now, let's differentiate $u(x)$ and $v(x)$.
Using the chain rule, the derivative of $u(x)$ is:

$$u'(x) = 3\left(x^3 + 2x\right)^2 \cdot (3x^2 + 2)$$

The derivative of $v(x)$ is:

$$v'(x) = 8(4x + 5)$$

Solution (Continued)

After differentiating $u(x)$ and $v(x)$, we apply the product rule:

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Substituting the derivatives:

$$f'(x) = \left[3\left(x^3 + 2x\right)^2 \cdot (3x^2 + 2)\right] \cdot (4x + 5)^2 + \left(x^3 + 2x\right)^3 \cdot 8\,(4x + 5)$$

The final derivative is:

$$f'(x) = 3\left(x^3 + 2x\right)^2 \cdot (3x^2 + 2) \cdot (4x + 5)^2 + \left(x^3 + 2x\right)^3 \cdot 8\,(4x + 5)$$

Example 2.12 (a)

Find the equation of the tangent line to the graph of $f(x) = x^2$ at $x_0 = 3$.
Solution:
To find the equation of the tangent line, we use the point-slope form:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is the point of tangency and m is the slope of the tangent line.
1. Find the point on the graph at $x_0 = 3$:

$$y_0 = f(x_0) = 3^2 = 9$$

So the point is $(3, 9)$.
2. Find the slope by differentiating $f(x) = x^2$:

$$f'(x) = 2x$$

Evaluate the derivative at $x_0 = 3$:

$$m = f'(3) = 2(3) = 6$$

So the slope is $m = 6$.

Solution(continued)

3. Write the equation of the tangent line:

$$y - 9 = 6(x - 3)$$

Simplify the equation:

$$y - 9 = 6x - 18 \quad \Rightarrow \quad y = 6x - 9$$

Therefore, the equation of the tangent line is:

$$y = 6x - 9$$

Example 2.12 (b)

Find the equation of the tangent line to the graph of $f(x) = \frac{x}{x^2+2}$ at $x_0 = 1$.

Solution:

We use the point-slope form of the tangent line equation:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is the point of tangency and m is the slope of the tangent line.

1. Find the point on the graph at $x_0 = 1$:

$$f(1) = \frac{1}{1^2+2} = \frac{1}{3}$$

So, the point is $(1, \frac{1}{3})$.

2. Find the slope by differentiating $f(x) = \frac{x}{x^2+2}$ using the quotient rule:

$$f'(x) = \frac{(x^2+2)(1) - x(2x)}{(x^2+2)^2} = \frac{-x^2+2}{(x^2+2)^2}$$

Evaluate the derivative at $x_0 = 1$:

$$f'(1) = \frac{-(1)^2+2}{(1^2+2)^2} = \frac{1}{9}$$

Solution (continued)

3. Write the equation of the tangent line: Using the point-slope form:

$$y - \frac{1}{3} = \frac{1}{9}(x - 1)$$

Simplifying:

$$y = \frac{1}{9}x + \frac{2}{9}$$

Therefore, the equation of the tangent line is:

$$y = \frac{1}{9}x + \frac{2}{9}$$