Linear Algebra-IV

Lecture 9

6 December 2024

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1 Matrix Algebra

Matrix Algebra

Definition

- A **matrix** is simply a rectangular array of numbers. So, any table of data is a matrix.
- The **size** of a matrix is indicated by the number of its rows and the number of its columns.
- A matrix with k rows and n columns is called a $k \times n$ ("*k by n*") matrix.
- The number in row i and column j is called the (i, j)th entry, and is often written a_{ij}

Matrix Algebra

Matrix

- The first array is called the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- When we add on a column to the coefficient matrix representing the constants on the right-hand side of the equations, we obtain the augmented matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Matrix Algebra

Matrix Operations

Addition

- One can add two matrices of the same size, which is to say, with the same number of rows and columns.
- Their sum is a new matrix of the same size as the two matrices being added. The (i, j)th entry of the sum matrix is simply the sum of the (i, j)th entries of the two matrices being added.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}.$$

Matrix Algebra

Example 1: Addition

$$\begin{pmatrix} 3 & 4 & 1 \\ 6 & 7 & 0 \\ -1 & 3 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 7 \\ 6 & 5 & 1 \\ -1 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 \\ 12 & 12 & 1 \\ -2 & 10 & 8 \end{pmatrix}.$$

Example 2: Addition

The matrices are:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}.$$

Matrix addition is defined only when the matrices have the same dimensions. Here:

- The first matrix is 2×3 (2 rows and 3 columns),
- The second matrix is 2×2 (2 rows and 2 columns).

Since the matrices have different dimensions, their sum is **undefined**.

Matrix Algebra

Matrix Operations

Subtraction

- Since $A - B$ is just shorthand for $A + (-B)$, we *subtract* matrices of the same size simply by subtracting their corresponding entries:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} - \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} - b_{k1} & \cdots & a_{kn} - b_{kn} \end{pmatrix}$$

Example: Subtraction

Consider the two matrices A and B of size 2×2:

$$A = \begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}.$$

To compute $A - B$, we subtract the corresponding entries of B from A:

$$A - B = \begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4-1 & 7-3 \\ 2-2 & 5-1 \end{pmatrix}.$$

Matrix Algebra

Matrix Operations

Scalar Multiplication

Matrices can be multiplied by ordinary numbers, which we also call **scalars**. This operation is called **scalar multiplication**. Implicitly we have already used this operation in defining $-A$, which is $(-1)A$. More generally, the product of the matrix A and the number r, denoted rA, is the matrix created by multiplying each entry of A by r:

$$r\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \cdots & ra_{kn} \end{pmatrix}$$

Within the class of $k \times n$ matrices, addition, subtraction, and scalar multiplication are all defined in the obvious way and act just as one would expect.

Matrix Algebra

Example: Scalar Multiplication

Consider the matrix A and the scalar $r = 3$:

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}, \quad r = 3.$$

To compute rA, we multiply each entry of the matrix A by r:

$$rA = 3 \cdot \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot (-2) \\ 3 \cdot 0 & 3 \cdot 4 \end{pmatrix}.$$

$$rA = \begin{pmatrix} 3 & -6 \\ 0 & 12 \end{pmatrix}.$$

Matrix Algebra

Matrix Operations

Matrix Multiplication

- Just as two numbers can be multiplied together, so can two matrices.
- But at this point matrix algebra becomes a little bit more complicated than the algebra for real numbers.
- There are two differences:
  1. Not all pairs of matrices can be multiplied together,
  2. the order in which matrices are multiplied can matter.

We can define the matrix product AB if and only if

number of columns of $A =$ *number of rows of* B.

Matrix Algebra

Matrix Operations

Matrix Multiplication

For the matrix product to exist, A must be $k \times m$ and B must be $m \times n$. To obtain the (i, j)th entry of AB, multiply the ith row of A and the jth column of B as follows:

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}.$$

The (i, j)th entry of the product AB is defined to be:

$$\sum_{h=1}^{m} a_{ih}b_{hj}.$$

Matrix Algebra

Example 1: Matrix Multiplication

The left matrix is 3×2, and the right matrix is 2×2. The result is a 3×2 matrix because the number of rows in the first matrix and the number of columns in the second matrix determine the result dimensions.

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}.$$

Matrix Algebra

Example 2: Matrix Multiplication

Note that in this case, the product taken in reverse order,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix},$$

is **not defined**.

- The first matrix is a 2×2 matrix.
- The second matrix is a 3×2 matrix.
- For matrix multiplication to be defined, the number of **columns** in the first matrix must equal the number of **rows** in the second matrix.
- The first matrix has 2 columns, but the second matrix has 3 rows. Since these numbers do not match, the product is **undefined**

Matrix Algebra

Matrix Operations

Matrix Multiplication

If A is $k \times m$ and B is $m \times n$, then the product AB will be $k \times n$. The product matrix AB inherits the number of its rows from A and the number of its columns from B:

$$\text{number of rows of } AB = \text{number of rows of } A;$$

$$\text{number of columns of } AB = \text{number of columns of } B;$$

$$(k \times m) \cdot (m \times n) = (k \times n).$$

Matrix Algebra

Example 3: Matrix Multiplication

Consider the matrices A and B, where:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}.$$

The matrix A is of dimension 2×3, and B is of dimension 3×2. The resulting product AB will be of dimension 2×2. The product AB are calculated as follows:

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}.$$

Matrix Algebra

Laws of Matrix Algebra

We can think of matrices as generalized numbers because matrix addition, subtraction, and multiplication obey most of the same laws that numbers do.

Associative Laws:

$$(A+B)+C = A+(B+C),$$
$$(AB)C = A(BC).$$

Commutative Law for Addition:

$$A+B = B+A.$$

Distributive Laws:

$$A(B+C) = AB+AC,$$
$$(A+B)C = AC+BC.$$

Matrix Algebra

Example for Associative Law: $(AB)C = A(BC)$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

Compute AB first:

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 2 & 1 \cdot 0 + 2 \cdot 1 \\ 0 \cdot 3 + 1 \cdot 2 & 0 \cdot 0 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then Compute $(AB)C$:

$$(AB)C = \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 \cdot 4 + 2 \cdot 0 & 7 \cdot 1 + 2 \cdot 2 \\ 2 \cdot 4 + 1 \cdot 0 & 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 28 & 11 \\ 8 & 4 \end{pmatrix}.$$

Matrix Algebra

Example for Associative Law: $(AB)C = A(BC)$ (cont.)

Compute BC first:

$$BC = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 0 \cdot 0 & 3 \cdot 1 + 0 \cdot 2 \\ 2 \cdot 4 + 1 \cdot 0 & 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 8 & 4 \end{pmatrix}.$$

Then compute $A(BC)$:

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 12 & 3 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 12 + 2 \cdot 8 & 1 \cdot 3 + 2 \cdot 4 \\ 0 \cdot 12 + 1 \cdot 8 & 0 \cdot 3 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 28 & 11 \\ 8 & 4 \end{pmatrix}.$$

Thus, $(AB)C = A(BC)$.

Matrix Algebra

Example for Distributive Law: $A(B+C) = AB + AC$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}.$$

Compute $B + C$:

$$B + C = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}.$$

Compute $A(B + C)$:

$$A(B+C) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 1 + 2 \cdot 5 \\ 3 \cdot 3 + 4 \cdot 4 & 3 \cdot 1 + 4 \cdot 5 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 25 & 23 \end{pmatrix}.$$

Compute AB and AC, then add:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 6 & 15 \end{pmatrix}, \quad AC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 19 & 8 \end{pmatrix}.$$

$$AB + AC = \begin{pmatrix} 2 & 7 \\ 6 & 15 \end{pmatrix} + \begin{pmatrix} 9 & 4 \\ 19 & 8 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 25 & 23 \end{pmatrix}.$$

Thus, $A(B + C) = AB + AC$.

Matrix Algebra

Example for Distributive Law: $(A+B)C = AC + BC$

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Calculate $A+B$:

$$A + B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 2 & 4 \end{pmatrix}.$$

Calculate $(A+B)C$:

$$(A+B)C = \begin{pmatrix} 5 & 5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 \cdot 2 + 5 \cdot 3 & 5 \cdot 1 + 5 \cdot 4 \\ 2 \cdot 2 + 4 \cdot 3 & 2 \cdot 1 + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 25 & 25 \\ 16 & 18 \end{pmatrix}.$$

Calculate AC and BC, then add:

$$AC = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 13 & 14 \end{pmatrix}, \quad BC = \begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 24 \\ 3 & 4 \end{pmatrix}.$$

$$AC + BC = \begin{pmatrix} 2 & 1 \\ 13 & 14 \end{pmatrix} + \begin{pmatrix} 23 & 24 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 25 & 25 \\ 16 & 18 \end{pmatrix}.$$

Thus, $(A+B)C = AC + BC$.

Matrix Algebra

Example for Commutative law for addition

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

Calculate $A + B$**:**

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$$

Calculate $B + A$**:**

$$B + A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$$

$$A + B = B + A.$$

Matrix Algebra

Commutative law for matrix multiplication

- The one important law which numbers satisfy but matrices do not, is the *commutative law for multiplication*.
- Although $ab = ba$ for all numbers a and b, it is not true that $AB = BA$ for matrices, *even when both products are defined*. But notice that even if both products exist, they need not be the same size.
- For example, if A is 2×3 and B is 3×2, then AB is 2×2 while BA is 3×3. Even if AB and BA have the same size, AB need not equal BA.

Matrix Algebra

Example: commutative law for multiplication

while

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

Matrix Algebra

Transpose

- The **transpose** of a $k \times n$ matrix A is the $n \times k$ matrix obtained by interchanging the rows and columns of A.
- This matrix is often written as A^T. The first row of A becomes the first column of A^T.
- The second row of A becomes the second column of A^T, and so on. Thus, the (i, j)-th entry of A becomes the (j, i)-th entry of A^T.

For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix}.$$

Matrix Algebra

Example: Transpose

Let A be a 2×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The transpose of A, denoted A^T, is obtained by interchanging its rows and columns. Therefore,

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

The first row of A, $(1, 2, 3)$ becomes the first column of A^T, and the second row of A, $(4, 5, 6)$ becomes the second column of A^T.

Matrix Algebra

Rules for Transpose

1. $$(A+B)^T = A^T + B^T,$$
2. $$(A-B)^T = A^T - B^T,$$
3. $$(A^T)^T = A,$$
4. $$(rA)^T = rA^T,$$

where A and B are $k \times n$ matrices and r is a scalar.

Matrix Algebra

Example for Transpose: Rule 1: $(A+B)^T = A^T + B^T$

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$$

Transpose both sides:

$$(A+B)^T = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B^T = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}.$$

$$A^T + B^T = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix}.$$

Thus, $(A+B)^T = A^T + B^T$.

Matrix Algebra

Example for Transpose: Rule 2: $(A - B)^T = A^T - B^T$

Let

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

$$A - B = \begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix}.$$

Transpose both sides:

$$(A - B)^T = \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix}.$$

Separately, compute A^T and B^T:

$$A^T = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Subtract the transposes:

$$A^T - B^T = \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix}.$$

Thus, $(A - B)^T = A^T - B^T$.

Matrix Algebra

Example for Transpose: Rule 3: $(A^T)^T = A$

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The transpose of A is:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Take the transpose of A^T:

$$(A^T)^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

Thus, $(A^T)^T = A$.

Matrix Algebra

Example for Transpose: Rule 4: $(rA)^T = rA^T$

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad r = 3.$$

First, calculate rA:

$$rA = 3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}.$$

Take the transpose:

$$(rA)^T = \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}.$$

Separately, calculate A^T and rA^T:

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad rA^T = 3 \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}.$$

Thus, $(rA)^T = rA^T$.

Matrix Algebra

Multiplication and Transpose

Theorem 8.1 Let A be a $k \times m$ matrix and B be an $m \times n$ matrix. Then,

$$(AB)^T = B^T A^T$$

Proof:

$$\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} \quad \text{(definition of transpose)},\\
&= \sum_h A_{jh} \cdot B_{hi} \quad \text{(definition of matrix multiplication)},\\
&= \sum_h (A^T)_{hj} \cdot (B^T)_{ih} \quad \text{(definition of transpose, twice)},\\
&= \sum_h (B^T)_{ih} \cdot (A^T)_{hj} \quad (a \cdot b = b \cdot a \text{ for scalars}),\\
&= (B^T A^T)_{ij} \quad \text{(definition of matrix multiplication)}.
\end{aligned}$$

Therefore, $(AB)^T = B^T A^T$.

Matrix Algebra

Example: Multiplication and Transpose

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

First, calculate AB:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Then, calculate $(AB)^T$:

$$(AB)^T = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}^T = \begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}.$$

Matrix Algebra

Example: Multiplication and Transpose (cont.)

First, calculate B^T and A^T:

$$B^T = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Then, calculate $B^T A^T$:

$$B^T A^T = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 + 7 \cdot 2 & 5 \cdot 3 + 7 \cdot 4 \\ 6 \cdot 1 + 8 \cdot 2 & 6 \cdot 3 + 8 \cdot 4 \end{pmatrix} = \begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}.$$

Notice that $(AB)^T = B^T A^T$, as both are:

$$\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}.$$