Linear Algebra-III

Lecture 8

29 November 2024

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Matrix

Definition

- The representation of a linear system by writing two rectangular arrays of its coefficients is called matrices.
- The first array is called the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- When we add on a column to the coefficient matrix representing the constants on the right-hand side of the equations, we obtain the augmented matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Matrix

Augmented Matrix

- For example, consider the system of equations:

$$x - 2y = 8$$
$$3x + y = 3$$

- The augmented matrix for this system is:

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 3 & 1 & 3 \end{bmatrix}$$

- For accounting purposes, it is often helpful to draw a vertical line just before the last column of the augmented matrix, where the = signs would naturally appear:

$$A = \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 3 & 1 & 3 \end{array}\right]$$

Matrix

Elementary Row Operations

Our three elementary row operations become **elementary row operations**:

1. Adding one equation to another.
2. Multiplying one equation by a non-zero number.
3. Exchanging the order of the equations.

The new augmented matrix will represent a system of linear equations which is equivalent to the system represented by the old augmented matrix.

To see this equivalence, first observe that each elementary row operation can be reversed . Clearly the interchanging of two rows or the multiplication of a row by nonzero scalar can be **reversed**.

Matrix

Elementary Row Operations

- Suppose we consider the row operation in which *-k* times the second row of the augmented matrix A is added to the first row of A. The new augmented matrix is:

$$B = \begin{bmatrix} a_{11} + ka_{21} & \dots & a_{1n} + ka_{2n} & b_1 + kb_2 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

- However, if we start with B and add *-k* times the second row to the first row, we will recover A. Thus the row operation can be reversed.

- Since elementary row operations correspond to the three operations of adding a multiple of one equation to another equation, multiplying both sides of an equation by the same scalar, and changing the order of the equations, any solution to the original system of equations will be a solution to the transformed system.
- Since these operations are reversible, any solution to the transformed system of equations will also be a solution to the original system. Consequently, the systems represented by matrices A and B have *identical solution sets*; they are equivalent.

Matrix

Row Echelon Form

A row of a matrix is said to have k leading zeros if the first k elements of the row are all zeros and the $(k+1)$th element of the row is not zero. With this terminology, a matrix is in **row echelon form** if each row has more leading zeros that the row preceding it.

Example: Row Echelon Form

The first row of the augmented matrix (C) has no leading zeros. The second row has one, and the third row has two. Since each row has more leading zeros than the previous row, matrix (C) is in **row echelon form**.

$$C = \begin{bmatrix} 1 & -0.4 & -0.3 & | & 130 \\ 0 & 0.8 & -0.2 & | & 100 \\ 0 & 0 & 0.7 & | & 210 \end{bmatrix}$$

Matrix

Example 1: Row Echelon Form

The matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 0 & 6 \\ 0 & 0 \end{pmatrix}$$

are in row echelon form. If a matrix in row echelon form has a row containing only zeros, then all the subsequent rows must contain only zeros.

Example 2: Row Echelon Form

The matrices

$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 7 \\ 9 & 0 \\ 0 & 2 \end{pmatrix}$$

are not in row echelon form.

Matrix

Pivot

It is natural that the first nonzero entry in each row of a matrix in row echelon form be called a **pivot**.

$$\begin{pmatrix} 1 & -0.4 & -0.3 & | & 130 \\ -0.2 & 0.88 & -0.14 & | & -74 \\ -0.5 & -0.2 & 0.95 & | & 95 \end{pmatrix},$$

The row echelon form of this matrix with various row operations is:

$$\begin{pmatrix} 1 & -0.4 & -0.3 & | & 130 \\ 0 & 0.8 & -0.2 & | & 100 \\ 0 & 0 & 0.7 & | & 210 \end{pmatrix}$$

The first rows of the above matrix is **pivot**

Matrix

Reduced Row Echelon Form

The row echelon form is the goal in the Gaussian elimination process. In Gauss-Jordan elimination, one wants to use row operations to reduce the matrix even further. First, multiply each row of the row echelon form by the reciprocal of the pivot in that row and create a new matrix all of whose pivots are 1s. Then, use these new pivots (starting with the 1 in the last row) to turn each nonzero entry *above* it (in the same column) into a zero.

Definition A row echelon matrix in which each pivot is a 1 and in which each column containing a pivot contains no other nonzero entries is said to be in **reduced row echelon form**.

Matrix

Example: Reduced Row Echelon Form

multiply the second row of matrix below by $1/0.8$ and the third row by $1/0.7$

$$\begin{pmatrix} 1 & -0.4 & -0.3 & | & 130 \\ 0 & 0.8 & -0.2 & | & 100 \\ 0 & 0 & 0.7 & | & 210 \end{pmatrix}$$

to achieve the matrix

$$\begin{pmatrix} 1 & -0.4 & -0.3 & | & 130 \\ 0 & 1 & -0.25 & | & 125 \\ 0 & 0 & 1 & | & 300 \end{pmatrix}$$

Matrix

Example: Reduced Row Echelon Form (cont.)

Then, use the pivot in row 3 to turn the entries -0.25 and -0.3 above it into zeros—first by adding 0.25 times row 3 to row 2 and then by adding 0.3 times row 3 to row 1. The result is

$$\begin{pmatrix} 1 & -0.4 & 0 & | & 220 \\ 0 & 1 & 0 & | & 200 \\ 0 & 0 & 1 & | & 300 \end{pmatrix}$$

Finally, use the pivot in row 2 to eliminate the nonzero entry above it by adding 0.4 times row 2 to row 1 to get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & | & 300 \\ 0 & 1 & 0 & | & 200 \\ 0 & 0 & 1 & | & 300 \end{pmatrix} \tag{15}$$

$$x_1 = 300, \quad x_2 = 200, \quad x_3 = 300.$$

Matrix (15) is in **reduced row echelon form**.

Matrix

Identity Matrix

The matrix whose diagonal elements (a_{ii}'s) are 1s and whose off-diagonal elements (a_{ij}'s with $i \neq j$) are all 0s is in row echelon form. This matrix arises frequently throughout linear algebra, and is called the **identity matrix** when the number of rows is the same as the number of columns.

Example: Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Matrix

Zero Matrix

The matrix each of whose elements is 0 is called **zero matrix** and is in *row echelon form*

Example: Zero Matrix

$$I = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Matrix

Rank of Matrix

Note that we say a row of a matrix is nonzero if and only if it contains at least one nonzero entry.

Definition: The **rank** of a matrix is the number of nonzero rows in its row echelon form.

The definition of rank requires that the rank is less than or equal to the number of rows of the coefficient matrix. Since each nonzero row in the row echelon form contains exactly one pivot, the rank is equal to the number of pivots.

Fact 7.1. Let A be the coefficient matrix and let $\hat{A}$ be the corresponding augmented matrix. Then,

(a) $\text{rank}\, A \leq \text{rank}\, \hat{A}$,

(b) $\text{rank}\, A \leq$ number of rows of A, and

(c) $\text{rank}\, A \leq$ number of columns of A.

Matrix

Example: Rank of Matrix

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

1. The first row remains as it is:

$$R_1 = \begin{pmatrix} 2 & -4 \end{pmatrix}.$$

2. To eliminate the first element of the second row, add $\frac{1}{2} \cdot R_1$ to R_2:

$$R_2 \to R_2 + \frac{1}{2} R_1.$$

Calculation:

$$R_2 = \begin{pmatrix} -1 & 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 & -4 \end{pmatrix} = \begin{pmatrix} -1+1 & 2-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Matrix

Example: Rank of Matrix (cont.)

The matrix now becomes:

$$\begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix}.$$

The rank of a matrix is the number of nonzero rows in its row echelon form. In this case:

- The first row is nonzero.
- The second row is a zero row.

Thus, the rank of the matrix A is:

$$\text{rank}(A) = 1.$$

Matrix

Example: Rank of Matrix

$$B = \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

1. Keep the first row as it is:

$$R_1 = \begin{pmatrix} 2 & -4 & 2 \end{pmatrix}.$$

2. Eliminate the first element of the second row by adding $\frac{1}{2} \cdot R_1$ to R_2:

$$R_2 \to R_2 + \frac{1}{2} R_1.$$

Calculation:

$$R_2 = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 & -4 & 2 \end{pmatrix} = \begin{pmatrix} -1+1 & 2-2 & 1+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}.$$

Matrix

Example: Rank of Matrix (cont.)

The matrix now becomes:

$$\begin{pmatrix} 2 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

The rank of a matrix is the number of nonzero rows in its row echelon form. Here:

- The first row is nonzero.
- The second row is also nonzero.

Thus, the rank of the matrix A is:

$$\text{rank}(A) = 2.$$