Linear Algebra-II

Lecture 7

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Systems of Linear Equation

Introduction

- Systems of Linear equations arise in two ways in economic theory. Some economic models have a natural linear structure.
- On the other hand, when the relationships among the variables under consideration are describes by a system of *nonlinear* equations, one takes the derivative of the equations to convert them to an approximating *linear* system.
- In this chapter we begin the study of systems of linear equations by describing techniques for solving such systems.

Systems of Linear Equation

Introduction

- The preferred solution technique-**Gaussian elimination** answers the fundamental questions about a given linear system: Does a solution exist, and if so, how many solutions are there?
- An implicit system is one in which the equations that describe the economic relationships under study have the *endogenous* and *exogenous* variables mixed in with each other on the same side of the equal signs.
- Linear Implicit Function Theorem uses linear algebra techniques *to quantify the effect of a change in the exogenous variables on the endogenous ones*.

Gaussian and Gaussian Jordan Elimination

Gaussian and Gaussian Jordan Elimination

- We begin our study of linear equations by considering the problem of solving linear systems of equations, such as

$$2x_1 + 3x_2 = 7 \quad \text{and} \quad x_1 - x_2 = 1 \tag{1}$$

or

$$x_1 + x_2 + x_3 = 5 \quad \text{and} \quad x_2 - x_3 = 0$$

- The general linear system of m equations in n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2}$$

Gaussian and Gaussian Jordan Elimination

Gaussian and Gaussian Jordan Elimination

- In this system, the a_{ij}'s are given real numbers; a_{ij} is the coefficient of the unknown x_j in the *i*th equation. A solution of system (2) is an *n*-tuple of real numbers $x_1, x_2, ..., x_n$ which satisfies each of the m equations (2). For example, $x_1 = 2, x_2 = 1$ solves the first system in (1), and $x_1 = 5, x_2 = 0, x_3 = 0$ solves the second.

- For a linear system such as (2), we are interested in the following three questions:
  1. Does a solution exist?
  2. How many solutions are there?
  3. Is there an efficient algorithm that computes actual solutions?

- There are essentially three ways of solving such systems:
  1. substitution
  2. elimination of variables
  3. matrix method

Substitution

Substitution

- Substitution is the method usually taught in beginning algebra. To use this method, solve one equation of system (2) for one variable, say x_n, in terms of the other variables in that equation.
- Substitute this expression for x_n into the other $m-1$ equations. The result is a new system of $m-1$ equations in the $n-1$ unknowns $x_1, \ldots, x_{n-1}$
- Use the earlier expressions of one variable in terms of the others to find all the x_i's.

Substitution

Example

The production process for a three-good economy is summarized by the input-output table:

Table 7.1: Input-Output Table $\begin{bmatrix} 0 & 0.40 & 0.30 \\ 0.2 & 0.12 & 0.14 \\ 0.5 & 0.20 & 0.05 \end{bmatrix}$

Suppose that there is an exogenous demand for 130 units of good 1, 74 units of good 2, and 95 units of good 3. How much will the economy have to produce to meet this demand?

Solution: Let x_i denote the amount of good i produced. "Supply equals demand" leads to the following system of equations:

$$\begin{aligned} x_1 &= 0x_1 + 0.4x_2 + 0.3x_3 + 130, \\ x_2 &= 0.2x_1 + 0.12x_2 + 0.14x_3 + 74, \\ x_3 &= 0.5x_1 + 0.2x_2 + 0.05x_3 + 95. \end{aligned}$$

Substitution

Example

Rearranging, we can write this as the system:

$$x_1 - 0.4x_2 - 0.3x_3 = 130, \quad (3a)$$
$$-0.2x_1 + 0.88x_2 - 0.14x_3 = 74, \quad (3b)$$
$$-0.5x_1 - 0.2x_2 + 0.95x_3 = 95. \quad (3c)$$

Solving equation (3a) for x_1 in terms of x_2 and x_3 yields:

$$x_1 = 0.4x_2 + 0.3x_3 + 130. \quad (4)$$

Substitute (4) into equations (3b) and (3c):

$$-0.2(0.4x_2 + 0.3x_3 + 130) + 0.88x_2 - 0.14x_3 = 74,$$
$$-0.5(0.4x_2 + 0.3x_3 + 130) - 0.2x_2 + 0.95x_3 = 95. \quad (5)$$

Substitution

Example (continued)

Which simplifies to:

$$0.8x_2 - 0.2x_3 = 100, \tag{5a}$$
$$-0.4x_2 + 0.8x_3 = 160. \tag{5b}$$

Now, use substitution to solve subsystem (5) by solving the first equation (5a) for x_2 in terms of x_3:

$$x_2 = 125 + 0.25x_3. \tag{6}$$

Substitute this expression into the second equation (5b):

$$-0.4(125 + 0.25x_3) + 0.8x_3 = 160,$$
$$x_3 = 300.$$

Substitution

Example (continued)

Substitute $x_3 = 300$ into (6) to compute:

$$x_2 = 125 + 0.25 \cdot 300 = 200.$$

Finally, substitute $x_2 = 200$ and $x_3 = 300$ into (4) to compute:

$$x_1 = 0.4 \cdot 200 + 0.3 \cdot 300 + 130 = 300.$$

Therefore, this economy needs to produce 300 units of good 1, 200 units of good 2, and 300 units of good 3 to meet the exogenous demands.

Elimination

Elimination

- This method is the most conducive to theoretical analysis.
- First, consider the simple system

$$x_1 - 2x_2 = 8 \tag{7a}$$
$$3x_1 + x_2 = 3 \tag{7b}$$

- We can eliminate the variable x_1 from this system multiplying equation (7a) by -3 to obtain $-3x_1 + 6x_2 = -24$ and adding this new equation to (7b). the results is

$$7x_2 = -21, x_2 = -3,$$

- To find x_1, we substitute $x_2 = -3$ back into (7b) or (7a) to compute that $x_1 = 2$.
- We chose to multiply equation (7a) by -3 precisely so that when we added the new equation to equation (7b), we would eliminate x_1 from the system. $x_1 = 2$

Elimination

Example

The production process for a three-good economy is summarized by the input-output table:

Table 7.1: Input-Output Table $\begin{bmatrix} 0 & 0.40 & 0.30 \\ 0.2 & 0.12 & 0.14 \\ 0.5 & 0.20 & 0.05 \end{bmatrix}$

$$x_1 - 0.4x_2 - 0.3x_3 = 130, \tag{8a}$$

$$-0.2x_1 + 0.88x_2 - 0.14x_3 = 74, \tag{8b}$$

$$-0.5x_1 - 0.2x_2 + 0.95x_3 = 95. \tag{8c}$$

Please use elimination method to find the solution.

Elimination

Example

- We first try to eliminate x_1 from the last two equations by adding to each of these equations a proper multiple of the first equation.
- To eliminate the $-0.2x_1$-term in (8b), we multiply (8a) by 0.2 and add this new equation to (8b). The result is the following calculation:

$$\begin{array}{r} 0.2x_1 - 0.08x_2 - 0.06x_3 = 26 \\ +(-0.2x_1 + 0.88x_2 - 0.14x_3) = 74 \\ \hline 0.8x_2 - 0.2x_3 = 100 \end{array}$$

Similarly, by adding 0.5 times (8a) to (8c), we obtain a new third equation:

$$-0.4x_2 + 0.8x_3 = 160$$

Elimination

Example

- Our equation system (8) has been transformed to the simpler system

$$
\begin{aligned}
1x_1 - 0.4x_2 - 0.3x_3 &= 130 &\quad (9a)\\
+0.8x_2 - 0.2x_3 &= 100 &\quad (9b)\\
-0.4x_2 + 0.8x_3 &= 160 &\quad (9c)
\end{aligned}
$$

- If one system of linear equations is derived from another by elementary equation operations, then both systems have the same solutions; that is; the systems are equivalent

- Having eliminated x_1 from the last two equations, we now want to eliminate x_2 from the last equation.

- We apply the elimination process to the system of two equations (9b) and (9c) in two unknowns.

Elimination

Example (cont.)

- Multiply (9b) by $1/2$ and add this new equation to (9c) to obtain the new system:

$$1x_1 - 0.4x_2 - 0.3x_3 = 130 \tag{10a}$$
$$+0.8x_2 - 0.2x_3 = 100 \tag{10b}$$
$$+0.7x_3 = 210 \tag{10c}$$

- Since each equation in (10) has one fewer variable than previous one, this system is particularly amenable to solution by substitution.
- Thus, $x_3 = 300$ from (10c). Substituting $x_3 = 300$ into (10b) gives $x_2 = 200$.
- Finally, substituting these two values into (10a) yields $x_1 = 300$

The method used in here is usually called *back substitution*

Elimination

Gaussian Elimination

This method of reducing a given system of equations by adding a multiple of one equation to another or by by interchanging equations until one reaches a system of the form (10) and then solving (10) via back substitution is called **Gaussian Elimination**.

The important characteristic of system (10) is that each equation contains fewer variables than the previous equation.

Elimination

Gauss Jordan Elimination

- We didn't use the operation of transforming an equation by simply multiplying it by a nonzero scalar.
- **Gauss Jordan Elimination** uses all three elementary equation operation. This method starts like Gaussian elimination, e.g., by transforming (89 to (10). After reaching system (10), multiply each equation in (10) by a scalar so that the first nonzero coefficient is 1:

$$x_1 - 0.4x_2 - 0.3x_3 = 130 \quad \text{(11a)}$$

$$x_2 - 0.25x_3 = 125 \quad \text{(11b)}$$

$$x_3 = 300 \quad \text{(11c)}$$

Elimination

Gauss Jordan Elimination

- Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11).
- For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of x_3 in (11b) and obtain $x_2 = 200$.
- Then add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$x_1 = 300 \tag{12a}$$
$$x_2 = 200 \tag{12b}$$
$$x_3 = 300 \tag{12c}$$

***Gauss-Jordan elimination is particularly useful in *developing the theory of linear systems*; Gauss elimination is usually more efficient in *solving actual linear systems*.