Linear Algebra-I

Lecture 6

22 November 2024

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Linear Systems

Introduction

- The analysis of many economic models reduces to the study of systems of equations.
- Some of the most frequently studied economic models are **linear models**.

Linear Equation

Typical linear equations are

$$x_1 + 2x_2 = 3 \text{ and } 2x_1 - 3x_2 = 8$$

They are called linear because their graphs are straight lines. In general, an equation is **linear** if it has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Linear Systems

Linear Equation

- The letters $a_1, ..., a_n$ and b stands for fixed numbers, such as 2,-3, and 8 in the second equation. These are called **parameters**.
- The letters $x_1, ..., x_n$ stand for **variables**
- The key feature of the general form of a linear equation is that each term of the equation contains at most one variable, and that variable appears only the first power rather than to the second, third, or some other power.

Linear Systems

Linear Equation

There are several reasons why it is natural to begin with systems of linear equations.

1. These are the most elementary equations that can arise. Linear algebra, the study of such systems, is one of the simpler branches of mathematics.
2. It requires no calculus and, at least in the beginning, very little familiarity with functions.
3. It builds on simple techniques, such as the solution of two linear equations in two unknowns via *substitution* and *elimination* of variables.
4. It also builds on the simple geometry of the plane and the cube, which is easy to visualize.
5. Linear systems have the added advantage that we can often calculate exact solutions to the equations.

Linear Systems

Example: Linear Equation

A company earns before-tax profits of 100,000 USD. It has agreed to contribute 10 percent of its after-tax profits to the Red Cross Relief Fund. It must pay a state tax of 5 percent of its profit (after the Red Cross Relief donation) and a federal tax of 40 percent of its profits (after the donation and state taxes are paid). How much does the company pay in state taxes, federal taxes, and Red Cross donation?

Let C, S, F represent the amounts of the charitable contribution, state tax, and federal tax, respectively. After-tax profits are $100,000 - (S + F)$;
so $C = 0.10 \cdot (100,000 - (S + F)$. We can write this as

$$C + 0.1S + 01.F = 10,000$$

putting all variables on one side. The statement that state tax is 5 percent of the profits net of donation becomes the equation $S = 0.05 \cdot (100,000 - C)$ or

$$0.05C + S = 5000$$

Federal taxes are 40 percent of the profit after deducting C and S; this relation is expressed by the equation $F = 0.40 \cdot [100,000 - (C + S)$, or

$$0.4C + 0.4S + F = 40000$$

Linear Systems

Example: Linear Equation (cont.)

We can summarize the payments to be made by the system of linear equations

$$C + 0.1S + 0.1F = 10000$$
$$0.05C + S = 5000$$
$$0.4C + 0.4S + F = 40000$$

There are a number of ways to solve this system. For example, you can solve the middle equation for S in terms of C, substitute this relation into the first and third equations in (1), and then easily solve the resulting system of two equations in two unknowns to compute

$$C = 5956, S = 4702, F = 35707$$

Solution to the Linear System

Steps 1 and 2: Setting up the system

We are given the following system of linear equations:

$$C + 0.1S + 0.1F = 10000 \tag{1}$$

$$0.05C + S = 5000 \tag{2}$$

$$0.4C + 0.4S + F = 40000 \tag{3}$$

Step 1: Solve for S from Equation (2)
From equation (2), we solve for S:

$$S = 5000 - 0.05C \tag{4}$$

Step 2: Substitute $S = 5000 - 0.05C$ into Equations (1) and (3)
Substituting equation (4) into equation (1):

$$C + 0.1(5000 - 0.05C) + 0.1F = 10000$$

Solution to the Linear System

Steps 1 and 2: Setting up the system

Simplifying:

$$C + 500 - 0.005C + 0.1F = 10000 \quad \text{and} \quad 0.995C + 0.1F = 9500 \tag{5}$$

We are given the following system of linear equations:
Now substitute $S = 5000 - 0.05C$ into equation (3):

$$0.4C + 0.4(5000 - 0.05C) + F = 40000$$

Simplifying:

$$0.4C + 2000 - 0.02C + F = 40000$$

$$0.38C + F = 38000 \tag{6}$$

Solution to the Linear System

Step 3: Solving the system

Step 3: Solve the system of two equations (5) and (6)
We now solve the system of two equations:

$$0.995C + 0.1F = 9500 \tag{5}$$

$$0.38C + F = 38000 \tag{6}$$

Solve equation (6) for F:

$$F = 38000 - 0.38C \tag{7}$$

Substitute equation (7) into equation (5):

$$0.995C + 0.1(38000 - 0.38C) = 9500$$

Simplifying:

$$0.995C + 3800 - 0.038C = 9500$$

$$0.957C + 3800 = 9500$$

$$0.957C = 5700 \quad \text{and} \quad C = 5700/0.957 \approx 5956$$

Solution to the linear System

Final Step

Step 4: Calculate S and F
Calculate S:
From equation (4):

$$S = 5000 - 0.05(5956) \approx 5000 - 297.8 = 4702$$

Calculate F:
From equation (7):

$$F = 38000 - 0.38(5956) \approx 38000 - 2262.28 = 35707$$

Final Solution:
Therefore, the solution to the system of equations is:

$$C \approx 5956, \quad S \approx 4702, \quad F \approx 35707$$