One Variable Calculus-V

Lecture 5

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Exponential Functions

Introduction

- In the last chapters, we dealt exclusively with relationships expressed by polynomial functions or by quotients of polynomial functions.
- However, in many economic models, the function which naturally models the growth of a given economic or financial variable over time has the independent variable t appearing as an $exponent$; for example, $f(t) = 2^t$

Example

- These exponential functions occur naturally, for example, as models for the amount of money in an interest-paying savings account for the amount of debt in fixed-mortgage account after t years

Exponential Functions

Definition

- First, we studied polynomials and rational functions and their generalizations of fractional and negative exponents - all function constructed by applying the usual arithmetic operations to the monomials ax^k.
- We now enlarge the class of functions under the study by including those functions in which the variable x appears as an $exponent$. These functions are naturally called **exponential functions**.

Example

- A simple example is $f(x) = 2^x$, a function whose domain is all the real numbers

Exponential Functions

Recall that:

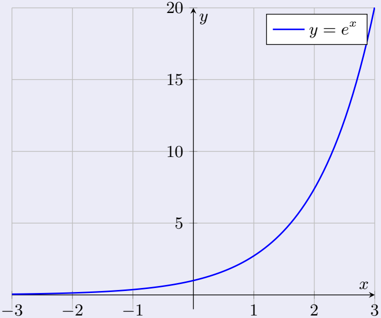
1. if x is a positive integer, 2^x means ”multiply 2 by itself x times”
2. if $x = 0$, $2^0 = 1$, by definition
3. if $x = 1/n$, $2^{1/n} = \sqrt[n]{2}$, the *n*th root of 2
4. if $x = m/n$, $2^{m/n} = (\sqrt[n]{2})^m$, the *m*th power of the *n*th root of 2
5. if x is a negative number, 2^x means $1/2^{|x|}$, the reciprocal of $2^{|x|}$

In these cases, the number 2 is called the **base** of the exponential function.

Exponential Functions

The Number e

- The number e plays the same fundamental role in economics and finance that the number π plays in geometry.
- In particular, the function $f(x) = e^x$ is called the **exponential function** and is frequently written as $exp(x)$.
- Since $2 < x < 3$, the graph of $exp(x) = e^x$ is shaped like



Exponential Functions

Theorem 5.1

As $n \to \infty$, the sequence $\left(1 + \frac{1}{n}\right)^n$ converges to a limit denoted by the symbol e.

Furthermore,

$$\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k$$

If one deposits A dollars in an account which pays annual interest at rate r compounded continuously, then after t years the account will grow to Ae^{rt} dollars.

Logarithm

Logarithm

- Consider a general exponential function, $y = a^x$, with base $a > 1$. Such an exponential function is strictly increasing:
$$x_1 > x_2 \implies a^{x_1} > a^{x_2}$$
In words, the more times you multiply a by itself, the bigger it gets.

- As we pointed out in Theorem 4.1, strictly increasing functions have natural inverses.

- Recall that the inverse of the function $y = f(x)$ is the function obtained by solving $y = f(x)$ for x in terms of y. For example, for $a > 0$, the inverse of the increasing linear function $f(x) = ax + b$ is the linear function $g(y) = (1/a)(y - b)$, which is computed by solving the equation $y = ax + b$ for x in terms of y:
$$y = ax + b \quad \Leftrightarrow \quad x = \frac{1}{a}(y - b)$$

- In a sense, the inverse g of f undoes the operation of f, so that
$$g(f(x)) = x$$

Logarithm

Logarithm

- We cannot compute the inverse of the increasing exponential function $f(x) = a^x$ explicitly because we can't solve $y = a^x$ for x in terms of y, as we did in previous slide.

- However, this inverse function is important enough that we give it name. We call it the **base** a **logarithm** and write

$$y = log_a(z) \quad \Leftrightarrow \quad a^y = z$$

- The **logarithm** of z, by definition, is the power to which one must raise a to yield z. It follows immediately from this definition that

$$a^{log_a(z)} = z \text{ and } log_a(a^z) = z$$

Logarithm

Example

1. The logarithm of 1000 is that power of 10 which yields 1000.
   Since $10^3 = 1000$, $\log_{10} 1000 = 3 \implies \log_{10} 10^3 = 3$
2. $\log_{10} 100,000 =?$
   Since $100,000 = 10^5 \implies \log_{10} 100,000 = 5$
3. $\log_{10} 10 =?$
   Since $10^1 = 10 \implies \log_{10} 10 = 1$
4. $\log_{10} 1 =?$
   Since $10^0 = 1 \implies \log_{10} 1 = 0$

Logarithms

Base e Logarithm

- Since the exponential function e^x has all the properties that 10^x has, it also has an inverse. Its inverse works the same way that $\log x$ does. Mirroring the fundamental role that e plays in applications, the inverse of e^x is called the **natural logarithm** function and is written as $\ln x$. Formally,

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

- $\ln x$ is the power to which one must raise e to get x. This definition can be summarized by the equations

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x$$

Logarithm

Example

1. $lne = 1$ since $e^1 = e$
2. $ln1 = 0$ since $e^0 = 1$
3. $ln40 = 3.688...$ since $e^{3.688...} = 40$

Properties of Exponential and Logarithm

Five Basic properties of Exponential Function

1. $a^r \cdot a^s = a^{r+s}$
2. $a^{-r} = 1/a^r$
3. $a^r/a^s = a^{r-s}$
4. $(a^r)^s = a^{r \cdot s}$
5. $a^0 = 1$

Properties of Exponential and Logarithm

The five properties of exponential functions are mirrored by five corresponding properties of the logarithmic functions:

1. $log(r \cdot s) = log(r) + log(s)$
2. $log(1/s) = -log(s)$
3. $log(r/s) = log(r) - log(s)$
4. $logr^s = (s)log(r)$
5. $log1 = 0$

The fifth property of logs follows directly from the the fifth property of a^x and the fact that a^x and log_a are inverses of each other.

Properties of Exponential and Logarithm

Example 1

To solve the equation $2^{5x} = 10$ for x, we take the logarithm of both sides:

$$\log(2^{5x}) = \log(10) \quad \text{or} \quad 5x \cdot \log(2) = 1$$

It follows that

$$x = \frac{1}{5\log(2)} \approx 0.6644$$

Properties of Exponential and Logarithm

Example 2

To solve $2e^{6x} = 18$, we follow these steps:

1. Isolate the exponential term:
$$e^{6x} = \frac{18}{2} = 9$$
2. Take the natural logarithm:
$$\ln(e^{6x}) = \ln(9)$$
3. Apply $\ln(a^b) = b\ln(a)$:
$$6x = \ln(9)$$
4. Solve for x:
$$x = \frac{\ln(9)}{6}$$
5. Further simplify $(9 = 3^2)$:
$$x = \frac{2\ln(3)}{6} = \frac{\ln(3)}{3}$$

Thus, the solution is:
$$x = \frac{\ln(9)}{6} \quad \text{or} \quad x = \frac{\ln(3)}{3}$$

Properties of Exponential and Logarithm

Derivatives of Exponential and Logarithm

Theorem 5.2: The functions e^x and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order. Their first derivatives are given by:

$$\text{(a)} \quad (e^x)' = e^x$$

$$\text{(b)} \quad (\ln x)' = \frac{1}{x}$$

If $u(x)$ is a differentiable function, then:

$$\text{(c)} \quad (e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$\text{(d)} \quad (\ln u(x))' = \frac{u'(x)}{u(x)} \text{ if } u(x) > 0$$

Properties of Exponential and Logarithm

Example 1

Find the derivative of e^{5x}

To differentiate e^{5x}, we use the chain rule. Recall that the derivative of $e^{u(x)}$ is $e^{u(x)} \cdot u'(x)$, where $u(x)$ is a differentiable function.

In this case, $u(x) = 5x$. The derivative of $5x$ is 5. So, applying the chain rule:

$$\frac{d}{dx}e^{5x} = e^{5x} \cdot \frac{d}{dx}(5x) = e^{5x} \cdot 5$$

Thus, the derivative is:

$$\frac{d}{dx}e^{5x} = 5e^{5x}$$

Therefore, the derivative of e^{5x} is $5e^{5x}$.

Properties of Exponential and Logarithm

Example 2

Find the derivative of $5e^{x^2}$

To differentiate e^{x^2}, we use the chain rule. Recall that the derivative of $e^{u(x)}$ is $e^{u(x)} \cdot u'(x)$, where $u(x)$ is a differentiable function.

In this case, $u(x) = x^2$. The derivative of x^2 is $2x$. So, applying the chain rule:

$$\frac{d}{dx} 5e^{x^2} = 5e^{x^2} \cdot \frac{d}{dx}(x^2) = 5e^{x^2} \cdot 2x$$

Thus, the derivative is:

$$\frac{d}{dx} e^{x^2} = 5 \cdot 2xe^{x^2}$$

Therefore, the derivative of e^{x^2} is $10xe^{x^2}$.

Properties of Exponential and Logarithm

Example 3

Find the derivative of $\ln(x^2)$
To differentiate $\ln(x^2)$, we use the chain rule for logarithms. Recall that if $u(x)$ is a differentiable function, then:

$$\frac{d}{dx}\ln(u(x)) = \frac{u'(x)}{u(x)}$$

In this case, we have $u(x) = x^2$. First, find the derivative of x^2:

$$u'(x) = \frac{d}{dx}(x^2) = 2x$$

Applying the chain rule:

$$\frac{d}{dx}\ln(x^2) = 2x \cdot \frac{1}{x^2}$$

Therefore, the derivative of $\ln(x^2)$ is:

$$\frac{d}{dx}\ln(x^2) = \frac{2}{x}$$

Properties of Exponential and Logarithm

Example 4

Find the derivative of $(\ln x)^2$

To differentiate $(\ln x)^2$, we apply the chain rule. The chain rule states that if $f(g(x))$ is a composition of functions, then:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Here, we let $u = \ln x$ (the "inside" function), so the function becomes u^2 (the "outside" function). The derivative of u^2 with respect to u is $2u$, and the derivative of $\ln x$ with respect to x is $\frac{1}{x}$. Thus, applying the chain rule:

$$\frac{d}{dx}(\ln x)^2 = 2 \cdot \ln x \cdot \frac{1}{x}$$

Therefore, the derivative is:

$$\frac{d}{dx}(\ln x)^2 = \frac{2\ln x}{x}$$

Hence, the derivative of $(\ln x)^2$ is $\frac{2\ln x}{x}$.

Properties of Exponential and Logarithm

To compute the derivative of the general exponential function $y = b^x$

Derivatives of Exponential and Logarithm

Theorem 5.3: For any fixed positive base b:

$$(b^x)' = \ln(b) \cdot b^x$$

Properties of Exponential and Logarithm

Example

Find the derivative of 10^x

To differentiate 10^x, we use the general rule for derivatives of exponential functions. Recall that for any constant base a, the derivative of a^x is:

$$\frac{d}{dx}a^x = \ln(a) \cdot a^x$$

In this case, $a = 10$. Therefore, the derivative of 10^x is:

$$\frac{d}{dx}10^x = \ln(10) \cdot 10^x$$

Hence, the derivative of 10^x is $\ln(10) \cdot 10^x$.