One Variable Calculus-IV

Lecture 4

18 October 2024

Index

Chain Rule

Definition

- Many economic situations involve chains of relationships between economic variables: variable A affects variable B, which in turn affects variable C.

Example

- For example, in a model of a firm, the amount of input used determines the amount of output produced, and the amount of output determines the firm's revenue.

- Revenue is a direct function of output and an indirect or *composite function* of input.

Chain Rule

- **Chain Rule** describes the derivative of composite function in terms of derivatives of its component functions, so that if the effect of a change in input on output is known and the effect of a change in output on revenue is known, the effect of a change in input on revenue can be computed

Chain Rule

Composite Function

- If f and h are two functions on R^1, the function formed by first applying function g to any number x and then applying function h to the result $g(x)$ is called the **composite of functions** g and h and is written as

$$f(x) = h(g(x)) \text{ or } f(x) = (h \circ g)(x)$$

- The function f is called the **composite** of functions h and g; we say that "*f is h composed with g*" and the "*f is g followed by h*"

Example

- If $g(x) = x^2$ and $h(x) = x + 4$ then
- $h(g(x))$ or $(h \circ g)(x)$: put x^2 in place of x in the $x + 4$

$$(h \circ g)(x) = x^2 + 4$$

Chain Rule

Composite Function

- When working with a composite function $f(x) = h(g(x))$, it is natural to call the first one applied (g in this case) the **inside function** and the second function one applied (h) in this case) the **outside function**

Composite Function

In the composition $(x^2 + 3x + 2)^7$, the inside function is $g(x) = x^2 + 3x + 2$ and the outside function is $h(z) = z^7$

Chain Rule

Composite Function

- The functions which describe a firm's behavior, such as its profit functionπ, are usually written as functions of a firm's profit on the amount of labor input L it uses, one must compose the profit function with firm's production function $y = f(L)$, the function which tells how much output y the firm can obtain from L units of labor input. The result is a function

$$P(L) = \pi(f(L)) = (\pi \circ f)(L)$$

- For example, if $\pi(y) = -y^4 + 6y^2 - 5$ and $f(L) = 5L^{\frac{2}{3}}$ then $P(L) = \pi(f(L))$
  = $-(5L^{\frac{2}{3}})^4 + 6(5L^{\frac{2}{3}})^2 - 5$
  = $-625L^{\frac{8}{3}} + 150L^{\frac{4}{3}} - 5$

Chain Rule

Chain Rule

Chain Rule is the derivative of the outside times the derivative of the inside, but one must remember that the derivative of the outside function is evaluated at the inside function.

$$\frac{d}{dx}h(g(x)) = h'(g(x)) \cdot g'(x)$$

Chain Rule

Chain Rule

- Let's apply the Chain Rule to compute the derivative of the composite function $P = \pi \circ f$.
- The outside function is $\pi(y) = -(y)^4 + 6(y)^2 - 5$
- the derivative of the outside function is $\pi'(y) = -4(y)^3 + 12(y)$
- and the derivative evaluated at the inside function $f(L) = 5L^{\frac{2}{3}}$ is
$$\pi'(f(L)) = -4(5L^{\frac{2}{3}})^3 + 12(5L^{\frac{2}{3}})$$
- On the other hand, the derivative of the inside function f is
$$f'(L) = \frac{10}{3}L^{\frac{-1}{3}}$$
- Multiplying these two expressions according to the Chain Rule yields
$$P'(L) = \frac{d}{dL}(\pi(f(L)) = \pi'(f(L)) * f'(L))$$
$$= (-4(5L^{\frac{2}{3}})^3 + 12(5L^{\frac{2}{3}})) * (\frac{10}{3}L^{\frac{-1}{3}})$$

Chain Rule

Chain Rule

- which after simplifying equals

$$(-4 * 125L^2 + 60L^{\frac{2}{3}}) * (\frac{10}{3}L^{\frac{-1}{3}}) = -\frac{500}{3}L^{\frac{5}{3}} + 200L^{\frac{1}{3}}$$

- Note that this agrees with what we compute by directly taking the derivative of the expression for composite function $P(L)$:

$$(-625L^{\frac{8}{3}} + 150L^{\frac{4}{3}} - 5)' = -\frac{500}{3}L^{\frac{5}{3}} + 200L^{\frac{1}{3}}$$

Inverse Function

The Chain Rule is one of the most powerful theorems in calculus, both in analyzing applications and in deriving other principles of calculus. As an illustration, it will be used in this section to derive the formula for the derivative of the inverse of a function when the derivative of the original function is known.

Definition

For any given function $f : E_1 \to \mathbb{R}^1$, where E_1, the domain of f, is a subset of $\mathbb{R}^1$, we say the function $g : E_2 \to \mathbb{R}^1$ is an **inverse** of f if

$$g(f(x)) = x \text{ for all } x \text{ in the domain } E_1 \text{ of } f \text{ and}$$

$$f(g(z)) = z \text{ for all } z \text{ in the domain } E_2 \text{ of } g$$

Inverse Function

Definition

1. *Theorem 4.1*
   A function f defined on an interval E in $\mathbb{R}^1$ has a well-defined inverse on the interval $f(E)$ if and only if f is monotonically increasing on all of E or monotonically decreasing on all of E.

2. *Theorem 4.2*
   A C^1 function f defined on an interval E in $\mathbb{R}^1$ is one-to-one and therefore invertible on E if $f'(x) > 0$ for all $x \in E$ or $f'(x) < 0$ for all $x \in E$.

Inverse Function

Example

- Consider the demand relationship between the market price p and the amount x that consumers are willing to consume at that price.
- If the demand function is given by the linear function

$$x = 3 - 2p \quad (1)$$

- then the inverse demand function is obtained by solving the previous linear function for p in terms of x:

$$p = \frac{1}{2}(3 - x) \quad (2)$$

- To see that the functions described by expressions$1)$ and 2 are inverses of each other, form their composition by substituting the expression 2 for p into 1:

$$x = 3 - 2(\frac{1}{2}(3 - x)) = 3 - (3 - x) = x$$

Inverse Function

Example

Find the inverse of the function $y = x^2$ for $x, y \geq 0$.

1. The original function:
$$y = x^2$$
2. Solve for x: To find the inverse, switch x and y and solve for y:
$$x = y^2$$
3. Taking the square root: Solve for y:
$$y = \sqrt{x}$$
4. The domain: The inverse function $y = \sqrt{x}$ is valid for $x \geq 0$ and $y \geq 0$.

Thus, the inverse of $y = x^2$ for $x, y \geq 0$ is:
$$y = \sqrt{x} \quad \text{for } x \geq 0,\ y \geq 0$$

Inverse Function

Example (cont.)

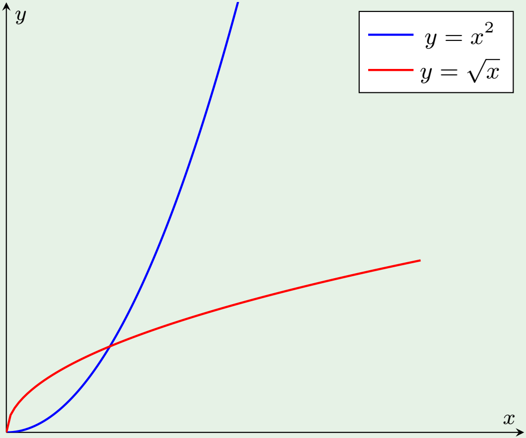

Figure 1: Graphs of $y = x^2$ and $y = \sqrt{x}$

The Derivative of the Inverse Function

Definition

If f is C^1 so that its graph has a smoothly varying tangent line, the graph of f^{-1} will also have a smoothly varying tangent line; that is, f^{-1} will be C^1 too.

Theorem 4.3 (Inverse Function Theorem)

Let f be a C^1 function defined on the interval I in $\mathbb{R}^1$. If $f'(x) \neq 0$ for all $x \in I$, then:

(a) f is invertible on I.

(b) Its inverse g is a C^1 function on the interval $f(I)$.

(c) For all z in the domain of the inverse function g:

$$g'(z) = \frac{1}{f'(g(z))}$$

The Derivative of the Inverse Function

Proof

- Since the graph of f^{-1} is the reflection of the graph of f across the diagonal line $y = x$, the graph of f^{-1} will have a well-defined tangent line everywhere be differentiable, if the graph of f does.
- Assuming that $g = f^{-1}$ is differentiable, we compute g' by first writing the inverse relation

$$f(g(z)) = z$$

- Now, take the derivative of both sides with respect to z, using the Chain Rule on the left side:

$$f'(g(z)) * g'(z) = 1$$
$$\text{or } g'(z) = \frac{1}{f'(g(z))}$$

The Derivative of the Inverse Function

Example 1

The inverse of $y = f(x) = mx$ is $x = g(y) = (\frac{1}{m})y$

Note that $g'(y) = \frac{1}{m} = \frac{1}{f'(x)}$

Example 2

The inverse of $y = f(x) = x^3$

- To find the inverse, switch x and y and solve for y: $x = y^3$
- Taking the cube root and solve for y: $y = \sqrt[3]{x}$
- The inverse function is $g(x) = \sqrt[3]{x}$.
- Find the derivative: $g'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{f'(g(x))}$

The Derivative of $x^{m/n}$

Theorem 4.4

We proved that the derivative of x^k is $kx^k - 1$ for any integer k. We will use *Theorem 4.3* and Chain Rule to show that this formula holds for *any rational number* $k = m/n$

For any positive integer n: $(x^{1/n})' = \frac{1}{n}x^{(1/n)-1}$

Proof The inverse of $y = x^{1/n}$ is $x = y^n$. By Theorem 4.3,

$$\begin{aligned}(x^{1/n})' &= \frac{1}{(y^n)'}, \text{ evaluated at } y = x^{1/n}, \\ &= \frac{1}{ny^{n-1}}, \text{ evaluated at } y = x^{1/n}, \\ &= \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1/n)-1}\end{aligned}$$

The Derivative of $x^{m/n}$

Theorem 4.5

For any positive integer m and n: $(x^{m/n})' = \frac{m}{n} x^{(m/n)-1}$

Proof Since $x^{m/n} = ((x^{1/n})^m$, we can apply the Chain Rule directly:

$$(x^{m/n})' = m(x^{1/n})^{m-1} * (x^{1/n})', \text{(by the Chain Rule)}$$

$$= mx^{(m-1)/n} * \frac{1}{n} x^{(1-n)/n}, \text{(by Theorem 4.4)}$$

$$= \frac{m}{n} x^{(m-n)/n} = \frac{m}{n} x^{(m/n)-1} \text{ (Simplifying)}$$