# One Variable Calculus-II

Lecture 2

4 October 2024

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# Computing Derivatives

## Theorem

Suppose that $k$ is an arbitrary constant and that $f$ and $g$ are differentiable functions of $x = x_0$.

1. $(x^k)' = kx^{k-1}$
2. $(kf)'(x_0) = kf'(x_0)$
3. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
4. $(f \cdot g)'(x_0) = f'(x_0) + g'(x_0)$
5. $(f/g)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$
6. $(f(x_0)^n)' = nf(x_0)^{n-1}f'(x_0)$

# Computing Derivatives

## Rule 1

$$f(x) = x^k \quad \Rightarrow \quad f'(x) = kx^{k-1}$$

## Examples

1. $f(x) = -x^8 + 5x^2 + 4x \quad \Rightarrow \quad f'(x) = -8x^7 + 10x + 4$
2. $f(x) = 5x^4 - 2x^2 \quad \Rightarrow \quad f'(x) = 20x^3 - 4x$
3. $f(x) = 3x^{2/3} + 3x^{-1} \quad \Rightarrow \quad f'(x) = 2x^{-1/3} - 3x^{-2}$

# Computing Derivatives

## Rule 2

$$(kf)(x) \quad \Rightarrow \quad (kf)'(x) = kf'(x)$$

## Example

**Example 1:** Let $f(x) = x^3$ and $k = 5$.

- The function we are differentiating is:

$$(kf)(x) = 5x^3$$

- The derivative of $f(x)$ is:

$$f'(x) = 3x^2$$

- Using the rule, we find:

$$(kf)'(x) = kf'(x) = 5 \cdot (3x^2) = 15x^2$$

# Computing Derivatives

## Rule 2

$$(kf)(x) \quad \Rightarrow \quad (kf)'(x) = kf'(x)$$

## Example

**Example 2:** Let $f(x) = 2x^4 + 3x^2 - 5$ and $k = -4$.

- The function we are differentiating is:

$$(kf)(x) = -4(2x^4 + 3x^2 - 5)$$

- First, calculate the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}(2x^4) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(-5) = 8x^3 + 6x$$

- Using the rule, we find:

$$(kf)'(x) = kf'(x) = -4(8x^3 + 6x) = -32x^3 - 24x$$

# Computing Derivatives

## Rule 3

$$(f \pm g)'(x_0) \quad = \quad f'(x_0) \pm g'(x_0)$$

## Examples

1. **Example 1:** Let $f(x) = x^7 + 3x^6 - 4x^2 + 5$
   - **Step 1:** Differentiate each term using the power rule:
     - Derivative of $x^7$ is $7x^6$
     - Derivative of $3x^6$ is $3 \cdot 6x^{6-1} = 18x^5$
     - Derivative of $-4x^2$ is $-4 \cdot 2x^{2-1} = -8x$
     - Derivative of constant $5$ is $0$
   - **Step 2:** Combine the derivatives:

$$f'(x) = 7x^6 + 18x^5 - 8x$$

# Computing Derivatives

## Rule 3

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$

## Examples (continued)

2. **Example 2:** Let $f(x) = 2x^5 - 3x^4 + 7x - 1$
   - **Step 1:** Differentiate each term using the power rule:
     - Derivative of $2x^5$ is $2 \cdot 5x^{5-1} = 10x^4$
     - Derivative of $-3x^4$ is $-3 \cdot 4x^{4-1} = -12x^3$
     - Derivative of $7x$ is $7$
     - Derivative of constant $-1$ is $0$
   - **Step 2:** Combine the derivatives:

$$f'(x) = 10x^4 - 12x^3 + 7$$

# Computing Derivatives

## Rule 4: Product Rule

$$(f \cdot g)'(x) \quad = \quad f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

## Example

**Example 1:** Use the product rule to differentiate the function:

$$(x^2 + 3x - 1)(x^4 - 8x)$$

- **Step 1:** Identify the Functions
  - $f(x) = x^2 + 3x - 1 \qquad g(x) = x^4 - 8x$
- **Step 2:** Calculate Derivatives
  - $f'(x) = 2x + 3 \qquad g'(x) = 4x^3 - 8$

# Computing Derivatives

## Example

- **Step 3:** Apply the Product Rule

$$(f \cdot g)'(x) = (2x+3)(x^4 - 8x) + (x^2 + 3x - 1)(4x^3 - 8)$$

- **Step 4:** Simplify the Expression (break down into smaller steps)

$$(2x+3)(x^4 - 8x) = 2x^5 + 3x^4 - 16x^2 - 24x$$
$$(x^2 + 3x - 1)(4x^3 - 8) = 4x^5 + 12x^4 - 4x^3 - 8x^2 - 24x + 8$$

Combining these results gives:

$$6x^5 + 15x^4 - 4x^3 - 24x^2 - 48x + 8$$

# Computing Derivatives

## Rule 4: Product Rule

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

## Example

**Example 2:** Use the product rule to differentiate the function:

$$h(x) = (2x^3 + x)(3x^2 - 5)$$

- **Step 1:** Identify the Functions
  - $f(x) = 2x^3 + x \qquad g(x) = 3x^2 - 5$
- **Step 2:** Calculate Derivatives
  - $f'(x) = 6x^2 + 1 \qquad g'(x) = 6x$

# Computing Derivatives

## Example (continued)

- **Step 3:** Apply the Product Rule

$$h'(x) = (6x^2 + 1)(3x^2 - 5) + (2x^3 + x)(6x)$$

- **Step 4:** Simplify the Expression

$$(6x^2 + 1)(3x^2 - 5) = 18x^4 - 30x^2 + 3x^2 - 5 = 18x^4 - 27x^2 - 5$$
$$(2x^3 + x)(6x) = 12x^4 + 6x^2$$

Combining these results gives:

$$h'(x) = 30x^4 - 21x^2 - 5$$

# Computing Derivatives

## Rule 5: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

## Example

**Example 1:** Use the Quotient Rule to differentiate the function:

$$h(x) = \frac{x^2 - 1}{x^2 + 1}$$

- **Step 1:** Identify the Functions
  - $f(x) = x^2 - 1 \qquad g(x) = x^2 + 1$
- **Step 2:** Calculate Derivatives
  - $f'(x) = 2x \qquad g'(x) = 2x$

# Computing Derivatives

## Example (continued)

- **Step 3:** Apply the Quotient Rule

$$h'(x) = \frac{(2x)(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2}$$

- **Step 4:** Simplify the Expression

$$\begin{aligned} h'(x) &= \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \\ &= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} \\ &= \frac{2x(2)}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

Therefore, the derivative is:

$$h'(x) = \frac{4x}{(x^2+1)^2}$$

# Computing Derivatives

## Rule 5: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

## Example

**Example 2:** Use the Quotient Rule to differentiate the function:

$$y(x) = \frac{2x^2 + 3x}{x^3 - 4}$$

- **Step 1:** Identify the Functions

$$f(x) = 2x^2 + 3x \quad g(x) = x^3 - 4$$

- **Step 2:** Calculate Derivatives

$$f'(x) = 4x + 3 \quad g'(x) = 3x^2$$

- **Step 3:** Apply the Quotient Rule

$$y'(x) = \frac{(4x + 3)(x^3 - 4) - (2x^2 + 3x)(3x^2)}{(x^3 - 4)^2}$$

# Computing Derivatives

## Example (continued)

- **Step 4:** Simplify the Expression

$$(4x+3)(x^3-4) = 4x^4+3x^3-16x-12$$
$$(2x^2+3x)(3x^2) = 6x^4+9x^3$$

Putting it all together gives:

$$y'(x) = \frac{(4x^4+3x^3-16x-12)-(6x^4+9x^3)}{(x^3-4)^2}$$

Combining these results gives:

$$y'(x) = \frac{-2x^4-6x^3-16x-12}{(x^3-4)^2}$$

# Computing Derivatives

## Rule 6: Power Rule

$$(f(x)^n)' = n \cdot (f(x))^{n-1} \cdot f'(x)$$

## Example

**Example 1:** Use the Power Rule to differentiate the function:

$$f(x) = (x^3 - 4x^2 + 1)^5$$

- **Step 1:** Identify the Functions

$$f(x) = x^3 - 4x^2 + 1 \quad n = 5$$

- **Step 2:** Calculate the Derivative

$$f'(x) = 3x^2 - 8x$$

$$f' = 5 \cdot (x^3 - 4x^2 + 1)^{5-1} \cdot (3x^2 - 8x)$$

- **Step 3:** Simplify the Expression

$$f'(x) = 5(x^3 - 4x^2 + 1)^4(3x^2 - 8x)$$

# Computing Derivatives

## Rule 6: Power Rule

$$(f(x)^n)' = n \cdot (f(x))^{n-1} \cdot f'(x)$$

## Example

**Example 2:** Use the Power Rule to differentiate the function:

$$f(x) = (2x+3)^4$$

- **Step 1:** Identify the Functions

$$f(x) = 2x+3 \quad n = 4$$

- **Step 2:** Calculate the Derivative

$$f'(x) = 2$$

$$f' = 4 \cdot (2x+3)^{4-1} \cdot f'(x)$$

- **Step 3:** Simplify the Expression

$$f'(x) = 8(2x+3)^3$$

# Computing Derivatives

## Differentiable Function

- A function of $f$ is differentiable at $x_0$, if, geometrically ist graph has a tangent line at $(x_0, f(x_0))$ or

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

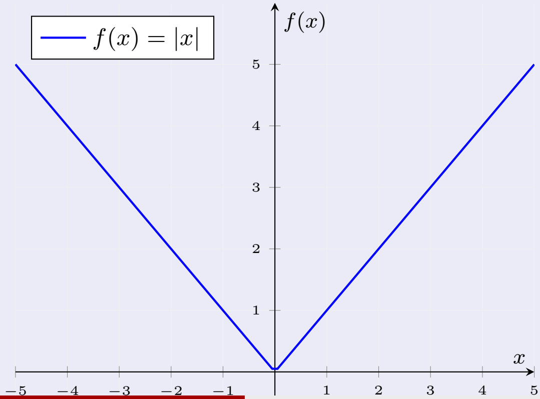
exists and is the same for every sequence $h$ which convergence to $0(zero)$.

- If a function is differentiable at every point $x_0$, we say that function is differentiable.

# Computing Derivatives

## A Non Differentiable Function

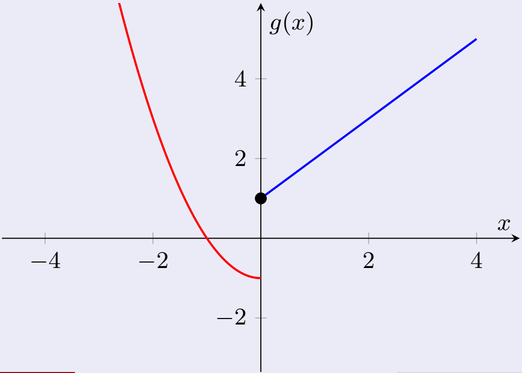
- As an example of a function which is not differentiable everywhere, consider the graph of absolute value function $f(x) = x$
- This graph has a sharper corner at the origin. There is no natural tangent line to this graph at $(0, 0)$. Since the graph of $|x|$ has no well defined tangent line at $x = 0$, the function $|x|$ is not differentiable at $x = 0$



# Computing Derivatives

## Continuous Function

- A function of $f$ is continuous if its graph has no breaks. Even though, it is not differentiable at $x = 0$, the function $f(x) = |x|$ is still continuous.
- $$g(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ x^2 - 1 & \text{if } x < 0 \end{cases}$$



# Computing Derivatives

## Continuous Function

1. Value of the function at $x = 0$:
   - According to the first function, when $x = 0$:
     $$g(0) = 0 + 1 = 1$$
   - Therefore, $g(0) = 1$.
2. Limit as $x$ approaches 0:
   - From the left (as $x$ approaches 0 from negative values):
     $$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x^2 - 1) = 0^2 - 1 = -1$$
   - From the right (as $x$ approaches 0 from positive values):
     $$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x + 1) = 0 + 1 = 1$$

# Computing Derivatives

## Continuous Function

3. About continuity:
   - For the function to be continuous at $x = 0$, the following must hold:
     $$\lim_{x \to 0} g(x) = g(0)$$
   - However, we found that:
     - $\lim_{x \to 0^-} g(x) = -1, \quad \lim_{x \to 0^+} g(x) = 1, \quad g(0) = 1$
4. Since the left-hand limit $(-1)$ does not equal the right-hand limit $(1)$, the overall limit does not exist at $x = 0$.

# Computing Derivatives

## Continuous Function

- It is not continuous at $x = 0$. In this case, we call the point $x = 0$ a discontinuity of $g$.
- It should be clear that the graph of a function can not have a tangent line at a point of discontinuity. In other words, in order for a function to be differentiable, it must at least be continuous.

# Computing Derivatives

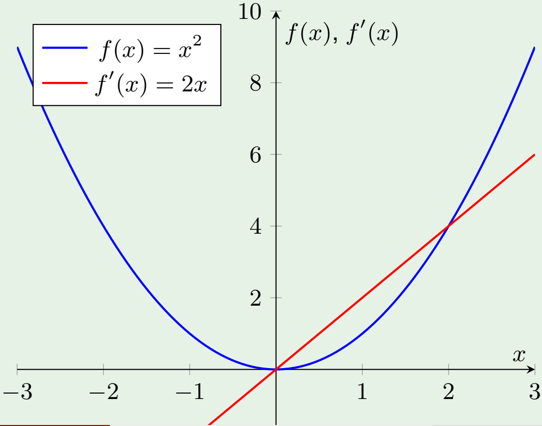
## Continuously Differentiable Function

- If $f$ is a differentiable function, its derivative $f'(x)$ is another function of $x$.
- It is the function which assigns to each point $x$ slope of the tangent line to the graph of $f$ at $x, f(x)$.
- Geometrically, the function $f'$ will be continuous if the tangent to the graph of f at $x, f(x)$ changes continuously as x changes.
- If $f'(x)$ is a continuous function of $x$, we say that $f$ is continuously differentiable function.

# Computing Derivatives

## Continuously Differentiable Function

- **Example:** Let $f(x) = x^2$. Then its derivative is $f'(x) = 2x$.
- Since $f'(x) = 2x$ is continuous for all $x$, we conclude that $f(x) = x^2$ is a continuously differentiable function.



# Computing Derivatives

## High Order Derivatives

- Let f be a continuously differentiable function $(C')$ on real numbers $(R^1)$.
- Since its derivative $f'(x)$ is a continuous function on $R^1$, we can ask whether or not the function $f'$ has a derivative at a point $x_0$.
- The derivative of $f'(x)$ at $x_0$ is called the **second derivative** of $f$ at $x_0$ and is written as:

$$f''(x_0) \quad \text{or} \quad \frac{d}{dx}\left(\frac{df}{dx}\right)(x_0) = \frac{d^2 f}{dx^2}$$

# Computing Derivatives

## High Order Derivatives

**Example 1** Find the high-order derivatives of the function:

$$f(x) = x^3 + 3x^2 + 3x + 1$$

- **Step 1:** Calculate the First Derivative

$$f'(x) = 3x^2 + 6x + 3 \quad f'(x) = 3 \cdot x^{3-1} + 3 \cdot 2 \cdot x^{2-1} + 3 \cdot 1 \cdot x^{1-1}$$

- **Step 2:** Calculate the Second Derivative

$$f''(x) = 6x + 6 \quad f''(x) = 6 \cdot x^1 + 6 \cdot x^0$$

- **Step 3:** Calculate the Third Derivative

$$f'''(x) = 6 \quad f'''(x) = 6 \cdot x^0$$

- **Step 4:** Calculate the Fourth Derivative

$$f^{(4)}(x) = 0 \quad f^{(4)}(x) = 0 \cdot x^{-1}$$

- **Conclusion:** Higher-order derivatives

$$f^{(n)}(x) = 0 \text{ for all } n \geq 4$$

# Computing Derivatives

## High Order Derivatives

**Example 2** Find the high-order derivatives of the function:

$$g(x) = 2x^4 + 5x^3 + x + 2$$

- **Step 1:** Calculate the First Derivative
$$g'(x) = 8x^3 + 15x^2 + 1 \quad g'(x) = 2 \cdot 4 \cdot x^{4-1} + 5 \cdot 3 \cdot x^{3-1} + 1 \cdot x^0$$
- **Step 2:** Calculate the Second Derivative
$$g''(x) = 24x^2 + 30x \quad g''(x) = 8 \cdot 3 \cdot x^2 + 15 \cdot 2 \cdot x^1$$
- **Step 3:** Calculate the Third Derivative
$$g'''(x) = 48x + 30 \quad g'''(x) = 24 \cdot x^1 + 30 \cdot x^0$$
- **Step 4:** Calculate the Fourth Derivative
$$g^{(4)}(x) = 48 \quad g^{(4)}(x) = 48 \cdot x^0$$
- **Step 5:** Calculate the Fifth Derivative
$$g^{(5)}(x) = 0 \quad g^{(5)}(x) = 0 \cdot x^{-1}$$

# Computing Derivatives

## Approximation by Differentials

- For a linear function, $f(x) = mx + b$, the der,vat,ve $f'(x) = m$ gives the slope of the graph of $f$ and measures the $rate of change or marginal change of f$ : **the increase in the value of f** for every unit increase in the value of $f$.
- For nonlinear function, we used the fact that the slope of the tangent line to the graph at $(x_0, f(x_0))$ is well approximated by the slope of the secant line through $(x_0, f(x_0))$ and nearby point $x_0 + h, f(x_0 + h)$ on the graph.

# Computing Derivatives

## Approximation by Differentials

- In symbols,

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx f'(x_0)$$

for $h$ small, where $\approx$ means ”is well approximated by” or ”is close in value to”.

- If we set $h = 1$ in the above equation, then it becomes

$$f(x_0 + 1) - f(x_0) \approx f'(x_0)$$

in words:

- The derivative of $f$ at $x_0$ is a good approximation to the marginal change of $f$ at $x_0$.

# Computing Derivatives

## Example 1: Approximation by Differentials

- Consider the production function $F(x) = \frac{1}{2}\sqrt{x}$.
- Suppose that the firm is currently using 100 units of labor input $x$, so that its output is 5 units.
- The derivative of the production function $F$ at $x = 100$:

$$F'(100) = \frac{1}{4}100^{-\frac{1}{2}} = \frac{1}{40} = 0.025,$$

- is a good measure of the **additional** output that can be achieved by hiring one more unit of labor, the **marginal product of labor.** The actual increase in output is $F(101) - F(100) \approx 0.02494$, which is pretty close to $0.025$.

# Computing Derivatives

## Example 2: Approximation by Differentials

- Consider the production function $G(k) = 3\sqrt{k}$.
- Suppose that the firm is currently using 64 units of capital input $k$, so that its output is 24 units.
- The derivative of the production function $G$ at $k = 64$:

$$G'(64) = \frac{3}{2} \cdot 64^{-\frac{1}{2}} = \frac{3}{16} = 0.1875,$$

- is a good measure of the **additional** output that can be achieved by increasing capital by one unit, the **marginal product of capital.** The actual increase in output is $G(65) - G(64) \approx 0.1871$, which is pretty close to $0.1875$.