Calculus of Several Variables

Lecture 12

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Multivariable Calculus in Economic Analysis

Introduction

- A primary goal in economic analysis is to understand how a change in one economic variable affects another.
- Chapter 3 demonstrated that one-variable calculus is the primary tool for understanding the effects of changes in economic relationships defined by functions of a single variable: $y = f(x)$.
- This chapter introduces multivariable calculus as the primary tool for understanding how variables affect others in economic relationships described by functions of several variables: $y = f(x_1, \ldots, x_n)$.

Partial Derivatives

Definition

- The change in one variable at a time, keeping all others constant (the variation brought by the change in one variable while others remain constant) is called the **partial derivative** of f with respect to x_i.
- The partial derivative is denoted by $\frac{\partial f}{\partial x_i}$, using Greek ∂ (delta) instead of Roman d.
- Other notations for $\frac{\partial f}{\partial x_i}$ include f_i, f_x, and $D_i f$.
- The derivative of a function f of one variable at x_0 is:

$$\frac{df}{dx}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- The partial derivative with respect to x_i of a function $f(x_1, \ldots, x_n)$ at $x^0 = (x_1^0, \ldots, x_n^0)$ is defined similarly.

Partial Derivative

Definition

- Let $f : \mathbb{R}^n \to \mathbb{R}$. For each variable x_i at each point $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ in the domain of f:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^0) = \lim_{h \to 0} \frac{f(x_1^0, \ldots, x_i^0 + h, \ldots, x_n^0) - f(x_1^0, \ldots, x_i^0, \ldots, x_n^0)}{h},$$

  if this limit exists.

- Only the ith variable changes; all other variables are treated as constants.

Partial Derivatives

Example

Consider the function $f(x, y) = 3x^2y^2 + 4xy^3 + 7y$. We calculate $\frac{\partial f}{\partial x}$ by treating y as a constant:

$$\frac{\partial}{\partial x}(3x^2y^2) = 2x \cdot 3y^2 = 6xy^2,$$
$$\frac{\partial}{\partial x}(4xy^3) = 4y^3,$$
$$\frac{\partial}{\partial x}(7y) = 0.$$

Putting this together:

$$\frac{\partial}{\partial x}(3x^2y^2 + 4xy^3 + 7y) = 6xy^2 + 4y^3.$$

Partial Derivatives

Example (cont.)

To calculate $\frac{\partial f}{\partial y}$, treat x as a constant:

$$\frac{\partial}{\partial y}(3x^2y^2) = 2y \cdot 3x^2 = 6x^2y,$$

$$\frac{\partial}{\partial y}(4xy^3) = 3y^2 \cdot 4x = 12xy^2,$$

$$\frac{\partial}{\partial y}(7y) = 7.$$

Combining these results:

$$\frac{\partial}{\partial y}(3x^2y^2 + 4xy^3 + 7y) = 6x^2y + 12xy^2 + 7.$$

Derivatives

Marginal Products

- For a function $y = f(x)$ of one variable, the derivative $f'(x)$ measures how a small change in x affects y:
$$\Delta y \approx f'(x)\Delta x.$$
- The same interpretation applies to functions of several variables.
- For example, let $Q = F(K, L)$ be a **production function**, which relates the output Q to the amounts of capital input K and labor input L.
- If the firm is currently using K^* units of capital and L^* units of labor to produce $Q^* = F(K^*, L^*)$ units of output, then the partial derivative:
$$\frac{\partial F}{\partial K}(K^*, L^*)$$
represents the rate at which output changes with respect to capital K, keeping L fixed at L^*.

Marginal Products

Marginal Product of Capital and Labor

- If capital increases by ΔK, then the output will increase by:

$$\Delta Q \approx \frac{\partial F}{\partial K}(K^*, L^*) \cdot \Delta K.$$

- Setting $\Delta K = 1$, we see that $\frac{\partial F}{\partial K}(K^*, L^*)$ estimates the change in output due to a one-unit increase in capital (with L fixed).
- $\frac{\partial F}{\partial K}(K^*, L^*)$ is called the **marginal product of capital** or **MPK**.
- Similarly, $\frac{\partial F}{\partial L}(K^*, L^*)$ is the rate at which output changes with respect to labor, with capital held fixed at K^*.
- $\frac{\partial F}{\partial L}(K^*, L^*)$ is called the **marginal product of labor** or **MPL**.

Marginal Product

Example: Cobb-Douglas Production Function

Consider the Cobb-Douglas production function:

$$Q = 4K^{3/4}L^{1/4}.$$

When $K = 10,000$ and $L = 625$, the output Q is:

$$Q = 4 \cdot 10,000^{3/4} \cdot 625^{1/4} = 4 \cdot 10^3 \cdot 5 = 20,000.$$

Calculating the partial derivatives:

$$\frac{\partial Q}{\partial K} = (4L^{1/4}) \left(\frac{3}{4} K^{-1/4} \right) = 3L^{1/4}K^{-1/4},$$

treating L as constant, and:

$$\frac{\partial Q}{\partial L} = (4K^{3/4}) \left(\frac{1}{4} L^{-3/4} \right) = K^{3/4}L^{-3/4},$$

treating K as constant.

Marginal Product

Example: Cobb-Douglas Production Function (cont.)

Furthermore:

$$\frac{\partial Q}{\partial K}(10,000,625) = \frac{3 \cdot 625^{1/4}}{10,000^{1/4}} = \frac{3 \cdot 5}{10} = 1.5,$$

and:

$$\frac{\partial Q}{\partial L}(10,000,625) = \frac{10,000^{3/4}}{625^{3/4}} = \frac{10^3}{5^3} = 8.$$

If L is held constant and K increases by ΔK, Q will increase approximately by

$1.5 \cdot \Delta K$.

For an increase in K of 10 units, we use (1) to estimate $Q(10,010,625)$ as:

$$20,000 + 1.5 \cdot 10 = 20,015.$$

This is a good approximation to $Q(10,010,625) = 20,014.998$ to three decimal places.
Similarly, a 2-unit decrease in L induces a $2 \cdot 8 = 16$-unit decrease in Q, so:

$$Q(10,000,623) = 20,000 + 8 \cdot (-2) = 19,984,$$

which approximates $Q(10,000,623) = 19,983.981$ to three decimal places.

Derivative

Elasticity

- Let $Q_1 = Q_1(P_1, P_2, I)$ represent the demand for good 1 as a function of the prices of goods 1 and 2, and income.
- The partial derivative $\frac{\partial Q_1}{\partial P_1}$ represents the rate of change of demand with respect to the price of good 1.
- If the price of good 1 rises by a small amount ΔP_1, the change in demand for good 1 can be approximated as:

$$\Delta Q_1 \approx \frac{\partial Q_1}{\partial P_1} \cdot \Delta P_1.$$

- In general, we expect $\frac{\partial Q_1}{\partial P_1}$ to be negative.
- As discussed earlier, the value of $\frac{\partial Q_1}{\partial P_1}$ depends on the units of measurement and is unsatisfactory as a measure of price sensitivity.

Derivative

Price Elasticity of Demand

- To remove dependency on units, economists measure the sensitivity of demand in percentage terms.
- The **price elasticity of demand**, ε_1, is defined as:

$$\varepsilon_1 = \frac{\%\text{change in demand}}{\%\text{change in own price}} = \frac{\Delta Q_1 / Q_1}{\Delta P_1 / P_1} = \frac{P_1}{Q_1} \cdot \frac{\Delta Q_1}{\Delta P_1}.$$

- Since:

$$\frac{\Delta Q_1}{\Delta P_1} = \frac{Q_1(P_1 + \Delta P_1) - Q_1(P_1)}{\Delta P_1} \approx \frac{\partial Q_1}{\partial P_1},$$

for small ΔP_1, this elasticity in calculus terms becomes:

$$\varepsilon_1 = \frac{P_1^* \cdot \frac{\partial Q_1}{\partial P_1}(P_1^*, P_2^*, I^*)}{Q_1(P_1^*, P_2^*, I^*)}.$$

Derivatives

Price Elasticity of Demand Elasticity (continued)

- The own price elasticity of demand, ε_1, is usually negative.
- If ε_1 lies between -1 and 0, good 1 is considered **inelastic**.
- If ε_1 lies between $-\infty$ and -1, good 1 is considered **elastic**, meaning a small percentage change in price results in a large percentage change in quantity demanded.

- To study the sensitivity of the demand for one good to price changes in *other goods*, economists use the **cross-price elasticity of demand**, ε_{Q_1,P_2}:

$$\varepsilon_{Q_1,P_2} = \frac{\%\text{change in demand for good 1}}{\%\text{change in price of good 2}} = \frac{\Delta Q_1/Q_1}{\Delta P_2/P_2}.$$

- The cross-price elasticity of demand, ε_{Q_1,P_2}, measures the sensitivity of demand for good 1 to price changes in good 2.

Derivatives

Cross-price Elasticity of Demand

- It is defined as:

$$\varepsilon_{Q_1,P_2} = \frac{\%\text{change in demand for good 1}}{\%\text{change in price of good 2}} = \frac{\Delta Q_1/Q_1}{\Delta P_2/P_2}.$$

- Using calculus, for small changes in P_2, this elasticity can be expressed as:

$$\varepsilon_{Q_1,P_2} = \frac{P_2^* \cdot \frac{\partial Q_1}{\partial P_2}(P_1^*, P_2^*, I^*)}{Q_1(P_1^*, P_2^*, I^*)}.$$

Derivative

Cross-Price Elasticity and Income Elasticity

- Cross-price elasticities can take on either sign:
  - If ε_{Q_1,P_2} and ε_{Q_2,P_1} are both positive, goods 1 and 2 are called **substitutes**.
  - If ε_{Q_1,P_2} and ε_{Q_2,P_1} are both negative, goods 1 and 2 are called **complements**.

- For substitutes, an increase in the price of good 1 increases demand for good 2.
- For complements, an increase in the price of one good decreases demand for both goods.

- To measure the sensitivity of demand to changes in income, economists study the **income elasticity of demand**, $\varepsilon_{Q_1,I}$:

$$\varepsilon_{Q_1,I} = \frac{\%\text{change in demand}}{\%\text{change in income}} = \frac{\Delta Q_1/Q_1}{\Delta I/I}.$$

- In calculus terms:

$$\varepsilon_{Q_1,I} = \frac{I}{Q_1} \cdot \frac{\partial Q_1}{\partial I},$$

evaluated at (P_1^*, P_2^*, I^*).

The Total Derivative

Introduction

Suppose we are interested in the behavior of a function $F(x, y)$ of two variables in the neighborhood of a given point (x^*, y^*). As noted earlier, if we hold y fixed at y^* and change x to $x^* + \Delta x$, then:

$$F(x^* + \Delta x, y^*) - F(x^*, y^*) \approx \frac{\partial F}{\partial x}(x^*, y^*)\Delta x. \tag{4}$$

If we hold x^* fixed and change y^* to $y^* + \Delta y$, then:

$$F(x^*, y^* + \Delta y) - F(x^*, y^*) \approx \frac{\partial F}{\partial y}(x^*, y^*)\Delta y. \tag{5}$$

The Total Derivative

Introduction

- If both x and y vary simultaneously, the combined effect is approximately the sum of the individual changes:

$$F(x^* + \Delta x, y^* + \Delta y) - F(x^*, y^*) \approx \frac{\partial F}{\partial x}(x^*, y^*)\Delta x + \frac{\partial F}{\partial y}(x^*, y^*)\Delta y. \qquad (6)$$

- Equation (6) can also be expressed in the form:

$$F(x^* + \Delta x, y^* + \Delta y) \approx F(x^*, y^*) + \frac{\partial F}{\partial x}(x^*, y^*)\Delta x + \frac{\partial F}{\partial y}(x^*, y^*)\Delta y. \qquad (7)$$

Derivative

Example: Total Derivative

- Consider the production function:

$$Q = F(K, L) = 4K^{3/4}L^{1/4},$$

  around the point $(K^*, L^*) = (10,000, 625)$.

- Using equation (4), we estimated:

$$F(10,010,625) \approx 20,015,$$

  and using equation (5), we estimated:

$$F(10,000,623) \approx 19,984.$$

- To consider the effect of **both changes**, we use equation (7) to estimate:

$$F(10,010,623) \approx F(10,000,625) + \frac{\partial F}{\partial K}(10,000,625) \cdot 10 + \frac{\partial F}{\partial L}(10,000,625) \cdot (-2).$$

- Substituting the values:

$$F(10,010,623) \approx 20,000 + 1.5 \cdot 10 + 8 \cdot (-2) = 19,999.$$