Linear Algebra-V

Lecture 10

13 December 2024

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Matrix Algebra

Special Kinds of Matrices

Square Matrix.
$k = n$, that is, an equal number of rows and columns.

Column Matrix.
$n = 1$, that is, one column. For example,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Row Matrix.
$k = 1$, that is, one row. For example,

$$\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \end{pmatrix}$$

Diagonal Matrix.
$k = n$ and $a_{ij} = 0$ for $i \neq j$, that is, a square matrix in which all nondiagonal entries are 0. For example,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Matrix Algebra

Special Kinds of Matrices

A **nonsingular matrix** is a square matrix whose rank equals the number of its rows (or columns). When such a matrix arises as a coefficient matrix in a system of linear equations, the system has one and only one solution.

Example: Nonsingular Matrix

Consider the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

To determine if A is nonsingular, we compute its determinant:

$$\det(A) = (2)(3) - (1)(1) = 6 - 1 = 5.$$

Since $\det(A) \neq 0$, A is nonsingular. Furthermore, A has full rank (rank = 2), which equals the number of rows and columns.
When used as a coefficient matrix in a system of linear equations, A ensures the system has one and only one solution.

Matrix Algebra

Algebra of Square matrices

- Within the class M_n of $n \times n$ (square) matrices, all the arithmetic operations defined so far can be used.
- The sum, difference, and product of two $n \times n$ matrices is $n \times n$. Even transposes of matrices in M_n are $n \times n$.
- The $n \times n$ identity matrix I is a true multiplicative identity in M_n in that:

  $$AI = IA = A \quad \text{for all } A \in M_n.$$

  The matrix I plays the role in M_n that the number 1 plays among the real numbers (M_1).
- Recall, however, that if A and B are in M_n, AB usually will not equal BA.

Matrix Algebra

Algebra of Square matrices

- Since we can add, subtract, and multiply square matrices, it is reasonable to ask if we can divide square matrices too.
- For numbers, dividing by a is the same as multiplying by $1/a = a^{-1}$, and a^{-1} makes sense as long as $a \neq 0$.
- To carry out this program for matrices (if we can), we need to make sense of A^{-1} for matrices in M_n. The number a^{-1} is defined to be that number b such that:

$$ab = ba = 1.$$

- The number b is called the inverse of the number a. We do the same for matrices in M_n.
- **Definition:** Let A be a matrix in M_n. The matrix B in M_n is an **inverse** for A if:

$$AB = BA = I.$$

  If the matrix B exists, we say that A is **invertible**.

Matrix Algebra

Theorem 8.6

If an $n \times n$ matrix A is invertible, then it is nonsingular, and the unique solution to the system of linear equations $Ax = b$ is:

$$x = A^{-1}b.$$

Proof: We want to show that if A is invertible, we can solve any system of equations $Ax = b$. Multiply each side of this system by A^{-1} to solve for x, as follows:

$$Ax = b,$$

$$A^{-1}(Ax) = A^{-1}b,$$
$$(A^{-1}A)x = A^{-1}b,$$
$$Ix = A^{-1}b,$$
$$x = A^{-1}b.$$

Thus, the solution $x = A^{-1}b$ is unique.

Matrix Algebra

Theorem 8.7

If an $n \times n$ matrix A is nonsingular, then it is invertible.

Theorem 8.8

The general 2×2 matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nonsingular (and therefore invertible) if and only if $ad - bc \neq 0$. Its inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Matrix Algebra

The Inverse of a Matrix

Definition:
The inverse of a matrix A is another matrix, denoted A^{-1}, such that:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

where I is the identity matrix.

- The matrix A must be **square**, i.e., it has the same number of rows and columns.
- The determinant of A, denoted $\det(A)$, must not be zero ($\det(A) \neq 0$).

For a 2×2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{if } ad - bc \neq 0.$$

Matrix Algebra

The Inverse of a Matrix

Methods for Larger Matrices:

- Gaussian Elimination
- Adjugate and Determinant:

$$A^{-1} = \frac{1}{\text{det}(A)} \cdot \text{adj}(A)$$

Note: A matrix with $\text{det}(A) = 0$ is called **singular** and does not have an inverse.

Matrix Algebra

Example 1: The Inverse of a Matrix

Inverse of a 2×2 Matrix

Let:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

The determinant of A is:

$$\det(A) = (2)(4) - (3)(1) = 8 - 3 = 5$$

The inverse is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

Matrix Algebra

Example 2: The Inverse of a Matrix

Inverse of a 3×3 Matrix (Gaussian Elimination)

Let:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

1. Augment the matrix B with the identity matrix:

$$[\ B \mid I\ ] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

2. Perform row operations to transform B into the identity matrix:
   1. Subtract $5 \times$ Row 1 from Row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

Matrix Algebra

Example 2: The Inverse of a Matrix

Inverse of a 3×3 Matrix (Gaussian Elimination)

2. Add $4 \times$ Row 2 to Row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

3. Subtract $3 \times$ Row 3 from Row 1:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Matrix Algebra

Example 2: The Inverse of a Matrix

Inverse of a 3×3 Matrix (Gaussian Elimination)

4. Subtract $4 \times$ Row 3 from Row 2:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

5. Subtract $2 \times$ Row 2 from Row 1:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

The right-hand side of the augmented matrix is the inverse:

$$B^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$