One Variable Calculus-I

Lecture 1

27 September 2024

Index

Foundation

Foundation

- A central goal of economic theory is to express and understand relationships between economic variables.
- These relationships are described mathematically by functions
- For instance, if we are interested in the effect of one economic variable (government spending) on one other economic variable (gross national product), we are led to the study of functions of a single variable.

Foundation

- The key information about these relationships between economic variables concerns how a change in one variable affects the other.
- When such relationships are expressed in terms of linear functions, the effect of a change in one variable on the other is captured by the ”slope” of the function.
- For more general nonlinear functions, the effect of this change is captured by the ”derivative” of this function.

Functions on Real Numbers

Function

- A function is simply a rule which assigns a number in R^1 to each number in R^1.

Example

- There is function which assigns to any number which is one unit larger.
- $f(x) = x + 1$, to the number 2 it assigns, the number 3 $f(2) = 3$
- $f(x) = x + 1$, to the number $-3/2$ it assigns the number $-1/2$ $f(-3/2) = -1/2$.

Functions on Real Numbers

Variables

- The input of the function x and the output of function y
- The input variable x is called the independent variable or in economic applications, the *exogenous variable*.
- The output variable y is called the dependent variable or in economic applications, the *endogenous variable*.

Example

- $y = x + 1$, x is the independent or exogenous variable
- $y = x + 1$, y is the dependent or endogenous variable

Functions on Real Numbers

Monomials

- Monomials are the simplest functions.
- Monomials can basically be written as $f(x) = ax^k$, k is called the *degree* of the monomial; a is called a *coefficient*.

Example

- $f(x) = 3x^4$
- 3 is the *coefficient* and 4 is the *degree* of monomial.

Functions on Real Numbers

Polynomials

- A functions which is formed by adding together monomials is called a *polynomial*.
- For any polynomial, the highest degree of any monomial that appears in it is called *degree* of the polynomial.

Example

- $f(x) = -x^7 + 3x^4 - 10x^2$
- the *degree* of polynomial is 7.

Functions on Real Numbers

Rational Functions

- The ratios of polynomials are called *rational functions*.

Example

- $y = \dfrac{x^2 + 1}{x - 1}$
- $y = \dfrac{x - 1}{x^3 + 3x + 2}$

Functions on Real Numbers

Exponential Functions

- The variable x appears as an exponent is called *exponential functions*.

Example

- $y = 10^x$
- $y = 5^x$

Functions on Real Numbers

Trigonometric Functions

- They are also known as the circular functions, since their values can be defined as ratios of the x and y coordinates of points on a circle of radius 1 that correspond to angles in standard positions.

Example

- $y = sinx$
- $y = cosx$

Functions on Real Numbers

Characteristics of Functions

- The basic geometric properties of a function are whether it is *increasing* or *decreasing* and the *locations* of its local and global *maxima* and global *minima*.

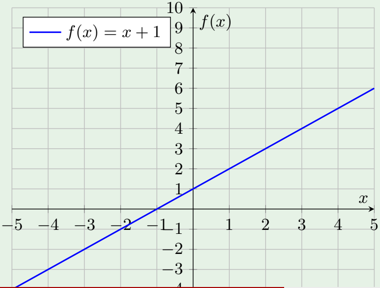
Functions on Real Numbers

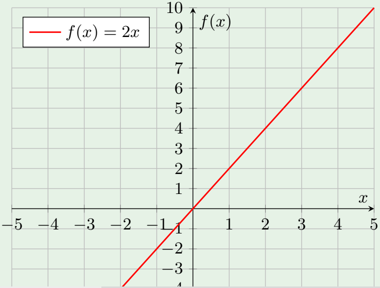
Increasing Function

- A function f is increasing if its graph moves upward from left to right
- $x_1 > x_2$ implies that $f(x_1) > f(x_2)$
- Increasing function says that there is a positive relationship between variables

Example: Increasing Functions

- $f(x) = x + 1$
- $f(x) = 2x$





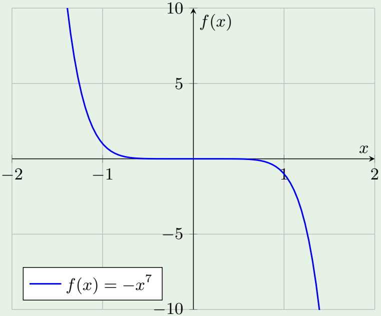
Functions on Real Numbers

Decreasing Function

- A function f is decreasing if its graph moves downward from left to right
- $x_1 > x_2$ implies that $f(x_1) < f(x_2)$
- Decreasing function says that there is a negative relationship between variables

Example

- $f(x) = -x^7$



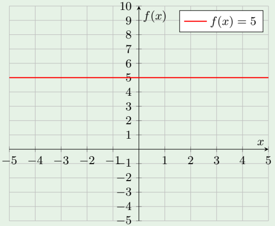
Functions on Real Numbers

Constant Function

- A constant function is a function whose (output) value is the same for every input value.
- Constant functions are linear functions whose graphs are horizontal lines in the coordinates.
- $x_1 > x_2$ implies that $f(x_1) = f(x_2)$

Example: Constant Function

- $f(x) = 5$



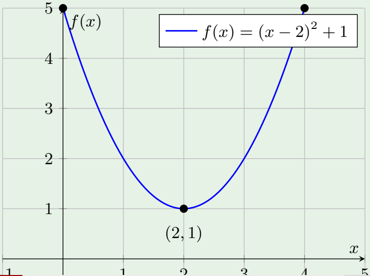
Functions on Real Numbers

Local (Relative) Minimum

- If a function f changes from decreasing to increasing at $[x_0]$, the graph of f turns upward around the point $((x_0), f(x_0))$.
- This implies that the graph of f lies above the point $((x_0), f(x_0))$.
- Such a point is called a *local or relative minimum*.

Example

- $f(x) = (x-2)^2 + 1$



Functions on Real Numbers

Global (Absolute) Minimum

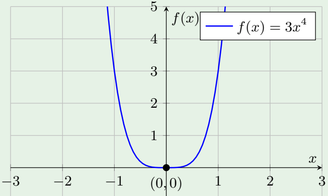
- If a graph of function f never lies below $((x_0), f(x_0))$, i.e, if

$$(f(x) \geq f(x_0)$$

  for all x, then $((x_0), f(x_0))$ is called a *global or absolute minimum* of f.
- This implies that the graph of f lies above the point $((x_0), f(x_0))$.
- Such a point is called a *global or absolute minimum*.

Example: Global (Absolute) Minimum

- $f(x) = 3x^4$



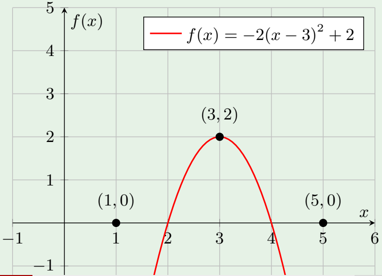
Functions on Real Numbers

Local (Relative) Maximum

- If a function g changes from increasing to decreasing at $[z_0]$, the graph of g turns downward around the point $((z_0), g(z_0))$.
- This implies that the graph of g lies below the point $((z_0), g(z_0))$.
- Such a point is called a *local or relative maximum*.

Example

- $f(x) = -2(x-3)^2 + 1$



Functions on Real Numbers

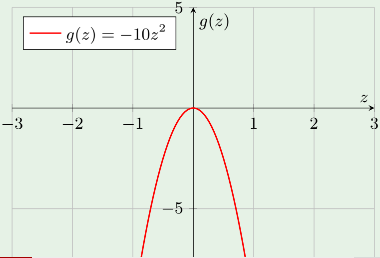
Global (Absolute) Maximum

- If a graph of function g never lies above $((z_0), g(z_0))$, i.e, if

$$(g(z) \leq g(z_0))$$

  for all z, then $((z_0), g(z_0))$ is called a *global or absolute maximum* of f.
- This implies that the graph of g lies below the point $((z_0), g(z_0))$.
- Such a point is called a *global or absolute maximum*.

Example: Global (Absolute) Maximum



Functions on Real Numbers

Domain

- Given a function f, the set of numbers x at which $f(x)$ is defined is called the *domain* of f.

Example

- $f(x) = 2x$ The domain of f is all of R^1) (Real Numbers)
- $g(x) = x + 1$ The domain of g is all of R^1) (Real Numbers)
- $m(x) = 3x^4$ The domain of m is all of R^1) (Real Numbers)
- $n(x) = -10x^7$ The domain of n is all of R^1) (Real Numbers)
- $h(x) = 1/(x^2 - 1)$ The domain of h is all x except of {-1,1}
- $j(x) = \sqrt{x-7}$ The domain of j is all $x \geq 7$

Functions on Real Numbers

Interval

- Given two real numbers a and b, the set of all numbers between a and b is called *interval*.
- If the endpoints a and b are excluded, the interval is called an *open interval* and written as

$$(a, b) = \{x \in \mathbb{R}^1 : a < x < b\}$$

- If the endpoints a and b are included, the interval is called a *closed interval* and written as

$$[a, b] = \{x \in \mathbb{R}^1 : a \leq x \leq b\}$$

- If only one endpoint a or b are included, the interval is called a *half-open interval* and written as

$$(a, b] = \{x \in \mathbb{R}^1 : a < x \leq b\}$$

- Infinite Interval

$$(a, \infty) = \{x \in \mathbb{R}^1 : a < x\}$$

Functions on Real Numbers

Linear Functions

- The simplest possible functions are the polynomials of degree 0.
- The simplest interesting functions are the polynomials of degree 1

Example

- $f(x) = b$, polynomials of degree 0: the constant functions
- $f(x) = mx + b$, polynomials of degree 1: linear functions
- $fx = mx + b$ is called linear function, because it is precisely the function its graph is straight line.

Functions on Real Numbers

Slope

- Interpreting the slope:

  Let (x_0, y_0) and $(x_1, y_1))$ be arbitrary points on a line R.
  The ratio $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ is called the **Slope**

- Slope is the rate of change and it measures how much value of the function, $f(x)$, increase (decrease) for each unit increase (decrease) in x.

- Two lines are parallel if and only if they have the same slope

Functions on Real Numbers

Slope

- If $C = F(q)$ is a linear cost function which gives the total cost (C) of manufacturing q units of output, then the slope of F measures the increase in total manufacturing cost due to the production of one more unit. In fact, it is the cost of making one more unit and it is called **Marginal Cost**.
- If $R = G(q)$ is a linear revenue function which gives the total revenue (R) from selling q units of output, then the slope of G measures the increase in total revenue due to the sale of one more unit. In fact, it represents the revenue gained from selling one additional unit and is called **Marginal Revenue**.

Functions on Real Numbers

Example

- The slope of the line joining the points (4,6) and (0,7). What is the slope?

$$m = \frac{7-6}{0-4} = -\frac{1}{4}$$

- Determine the slope of the line passing through the points (2, 3) and (5, 11).

y_1 = 11, y_0 = 3, x_1 = 5, x_0 = 2

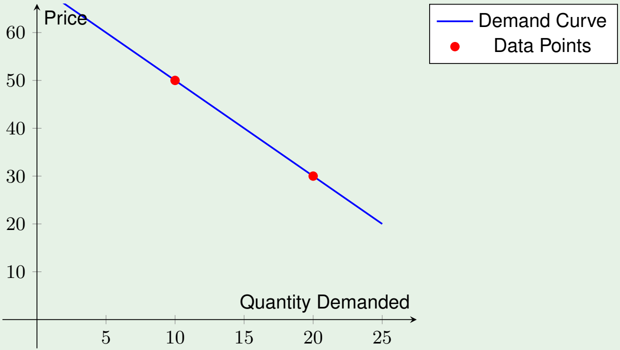
$$m = \frac{11-3}{5-2} = \frac{8}{3}$$

Example: Demand Curve Analysis

- The demand for a product is represented by the points $(x_0, y_0) = (10, 50)$ and $(x_1, y_1) = (20, 30)$, where x is the quantity demanded and y is the price. What is the slope of the demand curve?

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{30 - 50}{20 - 10} = \frac{-20}{10} = -2$$

- Since the slope of the demand curve is -2, this indicates that for each one unit increase in the quantity demanded (x), the price (y) decreases by 2 units.



Functions on Real Numbers

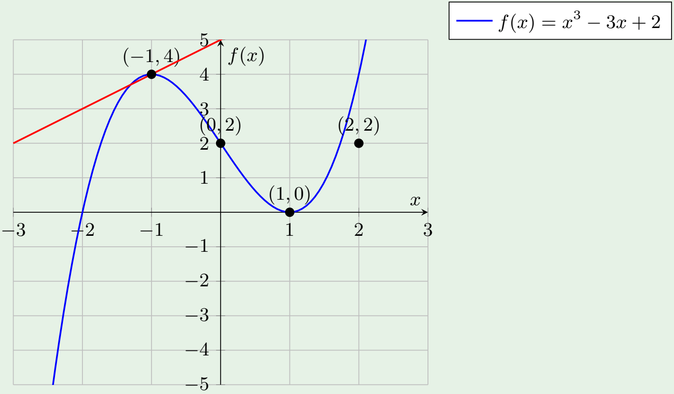
Nonlinear Functions

- Suppose that we are studying the nonlinear function $y = f(x)$ and that currently we are at the point $(x_0, f(x_0))$ on the graph of f. We want to measure the rate of change of f or the steepness of the graph of f when $x = x_0$.
- A natural solution to this problem is to draw the tangent line to the graph of f at x_0.
- Since the tangent line very close approximates the graph of f around $(x_0, f(x_0))$, it is a good proxy for the graph of f itself.
- Note that for nonlinear functions, unlike linear functions, the slope of the tangent line will vary from point to point.

Functions on Real Numbers

Nonlinear Function with Tangent Line

- Consider the function $f(x) = x^3 - 3x + 2$.



Functions on Real Numbers

Definition of Nonlinear Function

- We define the slope of a nonlinear function f at a point $(x_0, f(x_0))$ on its graph as the **slope of the tangent line** to the graph of f at that point.
- We call the slope of the tangent line to the graph of f at $(x_0, f(x_0))$ the **derivative of** f at x_0,
- we write it as

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}x_0$$

Functions on Real Numbers

Definition of Nonlinear Function

- Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$. The derivative of f at x_0 written

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx} x_0$$

is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$.

- Analytically,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists.

- When this limit doesn't exist, we say that the function f is differentiable at x_0 with derivative $f'(x_0)$